1 Planar graphs

1.1 More fun with faces

We developed this idea of a "face" of a planar projection of a graph G, which motivated the very useful result known as Euler's Formula: if a planar projection of a connected graph has v vertices, f faces, and e edges, then $v+f-e=2$. There is one useful immediate consequence of this theorem:

Corollary 1. *The number of faces of a planar projection connected graph G does not depend upon the choice of projection.*

We can develop several other properties of faces besides simply their number. Two interesting qualities of faces are their adjacencies and boundary lengths.

Definition 1. If the edges bounding a face F form a closed walk of length i, the face itself is said to have *length i*.

Note that edges separating the face from other faces will be counted once, while edges that jut into the face will be counted twice. This brings us to our second concept; namely, that two faces might share an edge.

Definition 2. If the edge e lies on the boundary of two separate faces f_1 and f_2, then f_1 and $f-2$ are said to be adjacent.

Definition 3. The *dual graph* of a planar projection of G is a graph H whose vertices are the faces of V, in which f_1 and f_2 are adjacent in H if they are adjacent in V.

The dual and face lengths are useful visualization tools, but have one grievous flaw: they aren't actually uniquely defined by the graph G, but only by its planar projection! Below are two planar projections of the same graph, with quite different duals ansd face-lengths.

For that reason, looking at the faces of G can be inherently dangerous if we treat them as if they were uniquely defined. However, to a certain extent we can make statements about them which are true regardless of projection. For instance:

Proposition 1. *If a planar graph G has faces $f_1, \ldots, f_k$ with respective lengths $\ell_1, \ldots, \ell_k$, then $\sum_{i=1}^{k} \ell_i = 2\|G\|$.*

Proof. Every edge is either a boundary between two faces, in which case it lies on the boundary walk of each face, contributing 1 towards the length of each and 2 towards the total, or lies wholly surrounded by a single face, in which case it appears twice on the boundary walk for a single face, contributing twice towards its length, and thus twice towards the total. Thus, each edge is accounted for twice in a census of all face-boundaries, so the above sum is twice the number of edges in G. □

Proposition 2. *In a planar connected graph G with 3 or more vertices, every face has length at least 3.*

Proof. If G is acyclic, it has a single face with boundary demonstrably of length $2|G| > 3$. Otherwise, every face has a nontrivial walk on its boundary, which could be reduced to a cycle by ignoring the edges which jut into the face. Since cycles have length of at least 3, the boundary walks have length of at least 3. □

Theorem 1. *A planar graph G with $n \geq 3$ vertices can have no more than $3n - 6$ edges.*

Proof. Let us denote the number of faces in G by k, and enumerating them in a particular planar projection, let us denote their lengths by $\ell_1, \ldots, \ell_k$. By the above results, we know that each $\ell_i \geq 3$ and that $\sum_{i=1}^{k} \ell_i = 2\|G\|$. Together these tell us that $3k \leq 2\|G\|$, so $k \leq \frac{2}{3}\|G\|$.

Now, by Euler's Theorem,

$$\begin{aligned} \|G\| + 2 &= |G| + k \\ \|G\| + 2 &\leq n + \frac{2}{3}\|G\| \\ \frac{1}{3}\|G\| &\leq n - 2 \\ \|G\| &\leq 3n - 6 \end{aligned}$$

□

In consequence of this, we have several useful results.

Corollary 2. *K_5 is nonplanar.*

Proof. K_5 has 5 vertices and 10 edges. The maximum number of edges a 5-vertex planar graph can have is $3 \cdot 5 - 6 = 9$. □

Corollary 3. *If G is planar, then $\delta(G) \leq 5$.*

Proof. If every vertex of G has degree 6 or more then $\|G\| = \frac{1}{2}\sum_{v \in V(G)} d(v) \geq \frac{1}{2}(6|G|) > 3|G| - 6$, violating a necessary condition for planarity. □

We can also use the same argument as above, in a rather strengthened special-case form, to show that $K_{3,3}$ is nonplanar.

Theorem 2. *$K_{3,3}$ is nonplanar.*

Proof. Let us suppose that $K_{3,3}$ had a planar projection. Since $K_{3,3}$ is bipartite, there are no closed walks on it of odd length, so the boundary of each face must be even. Since every face has boundary length of at least 3, the additional condition that boundary length is even requires that the boundaries have length of at least 4. We know that the sum of boundary length over all faces is $2\|K_3, 3\| = 18$; since each face has length at least 4, we knbow there can be no more than 4 faces in this projection. However, in order to satisfy Euler's formula, we would need $2 - |G| + \|G\| = 5$ faces. □

1.2 Characterization of planarity by substructures

K_5 and $K_{3,3}$ were long considered to be the most straightforward examples of nonplanarity, but it was only in the 20th century that they were revealed to in fact be the fundamental building blocks of planarity.

Definition 4. A *subdivision* H of a graph G is a graph formed by replacing all edges of G with internally disjoint paths in H.

It is fairly obvious that every subdivision of a nonplanar graph is nonplanar, and any graph with a nonplanar subgraph is nonplanar, so it is not the slightest bit surprising to see the following fact:

Proposition 3. *Any graph with a subdivision of K_5 or $K_{3,3}$ as a subgraph is nonplanar.*

The really surprising fact, however, is that this is a complete criterion for non-planarity:

Theorem 3 (Kuratowski '30)**.** *Every nonplanar graph has a subdivision of K_5 or $K_{3,3}$ as a subgraph*

The proof of Kuratowski's Theorem is messy enough that we shall not prove it at this time. To give an idea of the proof in outline: we attempt to find a minimal counterexample to Kuratowski's Theorem. It is easy to see that it must be at least 2-connected. Some further analysis of how a 2-connected counterexample might look allows us to dismiss the possibility of 2-connectedness, so our graph must be 3-connected. Now, assuming 3-connectedness (which necessitates several independent paths among the vertices, which require that subdivisions of certian highly structured graphs are present), the assumption that a graph does not contain subdivisions of the specific graphs K_5 and $K_{3,3}$ tells us enough about vertiex adjacencies to guarantee successful placement in a planar projection.

2 Vertex coloring

The most famous context in which planar graphs appear is that of graph coloring. Graph coloring deals with the extent to which vertices of a graph can be divided into different "colors" so that no two adjacent vertices are colored the same color.

Definition 5. A *k-coloring* of a graph G is a map $c : V(G) \to \{1, 2, \dots, k\}$ such that if $u \sim v$, then $c(u) \neq c(v)$. A graph is called *k-colorable* if it can be colored with k colors. The *chromatic number* $\chi(G)$ is equal to the least k such that G is k-colorable.

We may observe several things about chromatic numbers: every nontrivial graph has $\chi(G) \geq 2$, since even a single edge requires two differently-colored endpoints. Every bipartite graph (including trees and even cycles) has $\chi(G) = 2$, since the parts could be assigned different colors. The odd cycle C_{2n+1} has chromatic number 3, since it cannot be labeled easily with 2 colors in a round, but a third color avoids the problems caused by using only 2 colors. The complete graph K_n requires n colors, since each vertex must have a different color than every other vertex.

We can distill some of these basic results down into a few simple ideas:

- $\chi(G) \geq \omega(G)$
- $\chi(G) \leq \Delta(G) + 1$.

- A graph with chromatic number k is "k-partite"; that is, its vertices can be partitioned into classes $A_1, A_2, \dots, A_k$, so no edges are internal to a set.

The second of these can be somewhat improved in all but a few special cases.

Theorem 4 (Brooks '41)**.** *If G is a connected graph other than a complete graph or odd cycle, $\chi(G) \leq \Delta(G)$.*

Proof. We shall consider each of the vertices in order, coloring each with the lowest-numbered color we can legally still use at that vertex; such a coloring is called *greedy.* We will see that if our ordering is well-chosen, it need only use $\Delta(G)$ colors. Let us denote $k = \Delta(G)$, and consider $k \geq 3$ (if $k < 3$, we have a path or odd cycle).

If G is not k-regular, choose a vertex of degree less than k to be our last-visited one. Build a spanning tree crawling outwards from this vertex, assigning a visitation order so that children in the tree are earlier than their parents. Thus, when assigning each vertex a color, since it has fewer than k "children" in building the spanning tree, we greedily assign it a color k or less, so since we have a coloring of G using k colors, $\chi(G) \leq k$.

If G is k-regular, we have a couple of possibilities. One easy one is when $\kappa(G) = 1$. Then, G has a cut-vertex v and components G_1 and G_2 of $G - v$. $G_1 + v$ and $G_2 + v$ have k-colorings from the previous case (since v has edges into both G_1 and G_2, so neither of these subgraphs will have degree k at v). Once we have k-colorings of $G_1 + v$ and $G_2 + v$, we need only permute the colors of one or the other so they agree on v, giving a k-coloring of G.

So now we are left to consider 2-connected, k-regular graphs. We can still assemble spanning trees from a single vertex, so it is only a matter of making sure that the last vertex we assign a color to in our greedy method doesn't get "stuck". We can actually find an ordering meeting this criterion based on three vertices meeting certain adjacency and connectivity criteria, but need to consider two separate cases. After we pick an arbitrary vertex v, one of two things can happen:

Case I: $G - v$ is 2-connected. Then since G is not a complete graph and is k-regular, $k < |G| - 1$, so there are vertices not adjacent to v. Thus there is a vertex u at a distance 2 from v, and they must by the definition of distance have a mutual neighbor w. These three points have the property that $u \sim w \sim v$, $u \not\sim v$, and $G - \{u, v\}$ is connected.

Case II: $\kappa(G - v) = 1$. Then $G - v$ has a decomposition into 2 or more blocks $B_1, B_2, \dots, B_r$ with tree structure. At least two blocks are leaves in the tree, since every tree has at least two leaves. If the cutvertices for these leaves are removed from $G - v$, the resulting graph is necessarily disconnected; if these leaf-blocks did not have a non-cutvertex adjacent to v, the same would be true in G, contradicting G's 2-connectedness. Thus, there are two distinct blocks possessing vertices u and w adjacent to v. Since they lie in distinct blocks and are not cutvertices, u and w are distinct and nonadjacent; since $d(v) = k \geq 3$, v has another neighbor, so since removal of u and w does not disconnect their constituent blocks, they do not disconenct $G - v$, and v itself is connected through points other than u. Thus, we have three points such that $u \sim v \sim w$, $u \not\sim w$, and $G - \{u, w\}$ is connected.

In either case, we are guaranteed a triple lying in an induced P_3 such that removing the endpoints does not disconnect G. For simplicity, we will use the nomenclature from the second case, where u, v, and w appear in that order on the path. Let us order the vertices in the following way: put u and w first and second, and then produce a spanning tree on $G - \{u, w\}$, oriented so that v is the "root". We assign an order of $G - \{u, w\}$ by placing every child on the tree before its parents, so v

will be the last vertex to have color assigned, and each previous vertex can be guaranteed a color of k or less, using the spanning-tree arguments described earlier.

Now, since u is first, it is assigned color 1; since w is second and not adjacent to the first vertex, it is also assigned color 1. Every other vertex except v we have seen gets assigned a color of k or less. Now, v has k neighbors, and two of them (u and w, to wit) have the same color, so only $k-1$ colors are actually used among v's neighbors, so there is a color we can greedily assign to it which is $\leq k$. □

Brooks's Theorem is not much of an improvement over the extremely straightforward upper bound, but it is the only general statement which has actually been proven: there are a number of conjectures around suggesting that the chromatic number is low as long as there are reasonably few cliques, but there's been little progress towards proof.

2.1 Failures of the known bounds

We saw above that $\omega(G) \leq \chi(G) \leq \Delta(G)+1$. Whenever we have a bound on a parameter, it impels two questions: first, when is the bound sharp, that is to say, an equality? And on the other hand, how inaccurate can the bound be, that is, can the gap between the two sides of the inequality be arbitrarily large?

The first question is easily answered: $\omega(G) = \chi(G)$ when $G = K_n$, and also for many graphs in which a clique is the most complicated structure; likewise, $\chi(G) = \Delta(G) + 1$ when $G = K_n$, nad in fact Brooks' theorem guarantees that K_n and C_{2n+1} are the *only* graphs for which this bound is sharp.

The second question is of varying difficulty for the two bounds. The bound $\chi(G) \leq \Delta(G) + 1$ can be shown to have arbitrarily large gap by considering the star graph $G = K_{1,n}$: this graph, being bipartite (and furthermore, a tree), has chromatic number of 2 regardless of n, but since it is n-regular, increasing n allows it to have arbitrarily large maximum degree.

Proving that the gap between $\omega(G)$ and $\chi(G)$ can be arbitrarily large is a bit trickier. We shall specifically show that there are arbitrarily large graphs of chromatic number which are K_3-free.

Definition 6. The *Mycielskian* of G, denoted $M(G)$, is constructed as such: given $V(G) = \{v_1, \ldots, v_n\}$, we let $V(M(G)) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z\}$ such that if $v_i \sim v_j$, then $x_i \sim x_j$, $x_i \sim y_j$, and $y_i \sim x_j$. In addition, for all i, $y_i \sim z$.

Proposition 4. *If G is K_3-free, then so is $M(G)$.*

Proof. Let us consider all possible types of tripls in $M(G)$, and see that each does not contain a K_3. Since x_i and y_i are never adjacent, we need only consider triples with distinct indices. For distinct i, j, k, the triple $\{x_i, x_j, x_k\}$ is mutually adjacent iff $\{v_i, v_j, v_k\}$ is, which cannot be the case since G is K_3-free. The same is true for $\{x_i, y_j, x_k\}$. Any triple with two or more y vertices cannot be a K_3, since no y_i is adjacent to any y_j. Finally, a triple with z would need to contain two y vertices, an already eliminated possibility, since z is only adjacent to vertices of the form y_i. □

Proposition 5. $\chi(M(G)) = \chi(G) + 1$

Proof. First, it is easy to see that $\chi(G) + 1$ colors are sufficient: color G with $\chi(G)$ colors, and use the same color as appears on each v_i for x_i and y_i. It is clear from the internal edge structure among the x and y vertices of $M(G)$ that this coloring is valid; now, color z with color $\chi(G)+1$ to get a valid $(\chi(G)+1)$-coloring of all vertices of $M(G)$.

Now, to see that $\chi(G)+1$ colors are necessary, let us attempt a $\chi(G)$-coloring. Since the x vertices are an identical copy of G, all $\chi(G)$ colors are necessary and sufficient for coloring the x vertices; let us establish some such coloring to have been performed. We shall also see that all $\chi(G)$ colors are necessary among the y vertices. Let us suppose to the contrary that there is a coloring of the y vertices so that some color is unrepresented, and without loss of generality let us suppose the color not represented among the colors of y_i is color $\chi(G)$. For each x_i of color 1, the associated y_i is not of color $\chi(G)$ but is adjacent to all the same vertices as x_i, so x_i could be changed to the color of y_i and maintain propriety of the coloring. Performing this task at every x_i induces a coloring on the x_i vertices in $\chi(G)-1$ colors, contradicting our known chromatic number of G. Thus, every color in $\{1,2,\ldots,\chi(G)\}$ must in fact appear among the y_i, which leaves z uncolorable, so $M(G)$ has no $\chi(G)$-coloring. □

2.2 Coloring of planar graphs

The most well-known problem in graph coloring is surely the *map-coloring* problem, which asks: given a map depicting several contiguous regions, how many colors are necessary to paint the map so that no two regions sharing a border are the same color? The regions on a map can easily be associated with vertices of a planar graph (or vice versa) by means of the previously mentioned planar dual, which means this question is equivalent to asking the chromatic number of a planar graph. It is easy to find planar graphs with chromatic number up to 4 (for instance, K_4), but planar graphs of larger chromatic number cannot be found. For a very long time this was a conjecture universally thoguht true, and then:

Theorem 5 (Appel, Haken, and Koch '77)**.** *If G is planar, then $\chi(G) \leq 4$.*

This theorem was proven via a computer-enumeration of "reducible configurations"; that is to say, structures whose presence in a planar graph requiring 5 colors or more could be replaced with a smaller structure serving the same purpose. The Appel-Haken-Koch proof identified 1,936 such structures and showed that some such structure must be present in any planar graph of chromatic number 5. The existence of such structures is inconsistent with the existence of a minimal planar graph of chromatic number 5, since a reducible structure would allow any such graph to be reduced in size.

The proof of the Four-Color Theorem, as it is universally known, is far outside our abilities (or, indeed, anyone's) to read and appreciate in full, since it is a long computer-generated list of structures and provable facts about those structures not actually made for human appreciation. However, some of the underlying ideas can be appreciated in much simpler if slightly inferior bounds on planar graph colors.

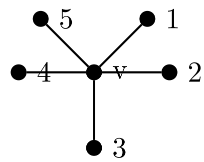
Proposition 6. *If G is planar, then $\chi(G) \leq 6$.*

Proof. Let us consider a minimal (in the sense of having as few vertices as possible) planar graph G such that $\chi(G) > 6$. By the edge-enumeration bounds derived from Euler's Formula, we know $\delta(G) \leq 5$. Thus, there is a $v \in G$ such that $d(v) \leq 5$. Since G is the minimal G with $\chi(G) > 6$, it follows that $\chi(G-v) \leq 6$. However, any 6-coloring of $G-v$ can be trivially extended to G, since v has only 5 neighbors and thus has a color which can be safely used. Thus, $\chi(G) \leq 6$, leading to a contradiction. □

The above choice of v to be a vertex with fewer than 6 neighbors is a good example of the aforementioned "reducible configurations" — it is an element whose removal from a graph of large chromatic number will not reduce the chromatic number.

Proposition 7. *If G is planar, then $\chi(G) \leq 5$.*

Proof. Let us consider a minimal planar G such that $\chi(G) > 5$. If $\delta(G) \leq 4$, we have the same proof as above with a very slight modification, so we must consider specifically the case where $\delta(G) = 5$. Specifically, we must consider the case of a vertex v with neighbors $u_1, \ldots, u_5$ such that among $u_1, \ldots, u_5$, all five colors are represented in any 5-coloring of $G - v$, since otherwise v could be easily colored consistently with $G - v$. WLOG, let us assume the vertices are numbered clockwise around v, and colored such that u_i has color i.



Looking just at the local neighborhood of v, we might wonder: why can't we just change u_1 to have color 3, and then give v the newly liberated color 1? This may be because u_1 already has a neighbor in color 3. So, we will branch out, determining the exact extent of obstruction of the color swap. Let S_{13} be the maximal set of connected vertices in color 1 and 3 containing u_1, so every vertex connected to u_1 by a path through only colors 1 and 3 is in S_{13}, and if $u \notin S_{13}$ is adjacent to a vertex in S_{13}, then u has color not equal to 1 or 3.

Since colors are fundamentally interchangable and no vertices of color 1 or 3 outside of S_{13} are adjacent to vertices in S_{13}, our valid coloring of $G - v$ will remain valid if we swap colors 1 and 3 in S_{13}. If $u_3 \notin S_{13}$, we will find that in this new coloring, v has no neighbors of color 1, admitting a trivial extension to a 5-coloring of G. Thus, since we claimed G is not 5-colorable, u_3 must be in S_{13}, so there is a path from u_1 to u_3 through vertices of color 1 and 3 only.

Using the same argument on u_2 and u_4, we find that there must be a path from u_2 to u_4 through vertices of color 2 and 4 only. But these cannot both be true on the planar projection presented: the u_1–u_3 path geometrically separates u_2 and u_4. □

You may wonder: why can't this argument be reduced to a simple proof of the Four-Color Theorem? After all, we only really used four of the points. However, if we had 4 colors appearing on 5 vertices adjacent to a single vertex, they could appear in many different configurations, and some of those configurations could prevent as easy an approach as we undertook above.

2.3 A Theoretical Basis for the Four-Color Theorem

The Four-Color Theorem and its attendant proof often leave people dissatisfied. Why *four* colors? What makes that number special, and is its specialness of a sort that is really that impossible to prove in a simple manner?

The question of how it is special in a way having to do with planarity is actually easily answered. The most straightforward example of a 5-colorable graph is K_5, which we *do* have a simple proof is nonplanar. Unfortunately, there are many 5-colorable graphs which do not possess a clear relationship to K_5, so there is no easy use of Kurotowski's theorem, or even this strengthening of Kuratowski's Theorem:

Theorem 6 (Wagner '37). *Any nonplanar graph G has $K_{3,3}$ or K_5 as a minor.*

Recall that a minor is a subgraph of the structure yielded by "merging" adjacent vertices and consolidating their neighbors as many times as necessary.

But even between K_5 minors and 5-chromatism there is a wide gulf. However, this idea permits us to make some interesting low-chromatic-number observations:

Proposition 8. *For $r = 1, 2, 3$, if $\chi(G) \geq r$, then G has a K_r minor.*

Proof. $\chi(G) = 1$ and presence of a K_1 minor are so trivial as to hardly bear mentioning. Presence of a K_2 minor is only slightly less trivial; $\chi(G) \geq 2$ if G contains any edges, and an edge is in itself a K_2 minor.

The $r = 3$ case is the first nontrivial one. If $\chi(G) \geq 3$, then G is nonbipartite and contains an odd cycle. K_3 is a minor of any cycle of length 3 or more, since edge contractions can reduce the cycle's length to 3. Thus every nonbipartite graph contains a K_3 minor.

The $r = 4$ case is quite nontrivial: Hadwiger showed in 1943 that a graph without a K_4 minor contains a vertex of degree ≤ 2; thus, by removing this vertex from a minimal graph of chromatic number 4 without a K_4 minor, and coloring it greedily on reintroduction, we can easily see that such a minimal example doesn't exist, and that every graph with a K_4 minor has chromatic number of at elast 4. □

This pattern strondly suggests a pattern of necessary (if not sufficient; note that C_4 has K_3 as a minor, but $\chi(C_4) = 2$) conditions for a graph to have high chromatic number, but it has not been proven.

Conjecture 1 (Hadwiger '43). *If $\chi(G) \geq r$, then G has a K_r minor.*

The $r = 5$ case of this conjecture is slightly stronger than the 4-color theorem, so it is highly doubtful that it has an elegant or even readable proof for the case or arbitrary r. It has been verified for $r \leq 6$: the proofs of the cases of $r = 5, 6$ in fact invoke the Four-Color Theorem.