1 Hamiltonian properties

1.1 Hamiltonian Cycles

Last time we saw this generalization of Dirac's result, which we shall prove now.

Proposition 1 (Ore '60)**.** *For a graph G with nonadjacent vertices u and v such that $d(u)+d(v) \geq |G|$, it follows that G is Hamiltonian if and only if $G+e$ is Hamiltonian, for $e = \{u, v\}$.*

Proof. In one direction this implication is trivial: if G is Hamiltonian, then clearly $G+e$ is Hamiltonian. To prove the other direction, let $G+e$ contain a Hamiltonian cycle C. If the Hamiltonian cycle doesn't use e, then C lies in G, so that G is Hamiltonian as well. If, however, e is in C, we do not immediately see a Hamiltonian cycle in G: instead, we have a Hamiltonian *path* $C-e$ in G.

Let us denote the vertices of this path $C-e$ by $u = v_0 \sim v_1 \sim \cdots \sim v_k = v$ in G. Since this path visits every vertex of G, we know that every neighbor of u and every neighbor of v is in the path, and since $u \not\sim v$, we know every neighbor of u and every neighbor of v is in $\{v_1, \ldots, v_{k-1}\}$. Let us consider the set A of neighbors of u, which clearly is a subset of $\{v_1, \ldots, v_{k-1}\}$ of size $d(u)$. Now consider the set B of *successors* of v's neighbors; that is to say, points v_i such that $v_{i-1} \sim v$; clearly this will be a subset of $\{v_1, \ldots, v_k\}$ of size $d(v)$. We now have two subsets of $\{v_1, \ldots, v_k\}$ of sizes $d(u)$ and $d(v)$; since $d(u)+d(v) \geq |G| > k$, these two sets must overlap, so there is some i such that both $u \sim v_i$ and $v_{i-1} \sim v$. Using this value, we can produce a Hamiltonian cycle in G:

$$u \sim v_1 \sim \cdots \sim v_{i-1} \sim v \sim v_{k-1} \sim v_{k-2} \sim \cdots \sim v_i \sim v_0$$

□

In consequence of this, we may add edges between any two non-adjacent vertices of total degree $|G|$ without affecting Hamiltonicity.

Definition 1. The *Hamiltonian closure* $\mathcal{C}(G)$ of a graph G is the graph resulting from adding edges between nonadjacent vertices of degree sum $|G|$ until it is impossible to do so any further. A graph is *closed* if for every pair u and v of vertices, either $d(u)+d(v) < |G|$ or u and v are adjacent.

The definition above contains a hidden assumption: namely, that the procedure described has a unique result regardless of choice of vertices.

Proposition 2. *The result of the Hamiltonian-closure procedure is the same regardless of order of edge-addition choice.*

Proof. Consider a closure which adds specific edges $e_1, \ldots, e_k$. We will prove by induction that any order of edge-addition must add the edge e_i for each i. For the base case, note that if $e_i = \{u_i, v_i\}$, then $d_G(u_1)+d_G(v_1) \geq |G|$; since adding edges can only increase degree, this inequality will remain true at every stage of edge-addition; thus it will become impossible to add more edges only after e_1 is added, so every edge-addition order will add e_1 eventually.

For the inductive step, let G' represent $G + e_1 + e_2 + \cdots + e_{i-1}$. We know by the definition of the closure procedure that $d_{G'}(u_i) + d_{G'}(v_i) \geq |G|$. Now, let us consider an arbitrary order of edge-addition. By our inductive hypothesis, this procedure will, at some point, add the edges $e_1, \ldots, e_{i-1}$. Thus, by the time *any* Hamiltonian-closure procedure is finished, our graph will contain all the edges of G', so our graph will have degree-sum exceeding $|G|$ between u_i and v_i, so, since a full closure procedure adds every edge between points of total degree $|G|$, the edge between u_i and v_i must be in this closure. □

Since the closure is the result of repeated applciations of the procedure mentioned in Ore's lemma, it clearly preserves Hamiltonicity:

Theorem 1 (Bondy and Chvátal, '76)**.** *G is Hamiltonian if and only if $\mathcal{C}(G)$ is closed.*

And in consequence, we do not, in principle, need to assess Hamiltonicity of every graph — just the closed ones! If we had a good test for Hamiltonicity of closed graphs, testing Hamiltonicity of arbitrary graphs would be easy by takign their closures and testing those. Unfortunately, the question of which graphs are Hamiltonian does not seem to become significantly easier as a result of limiting the scope to closed graphs.

However, the closure procedure has a somewhat cumulative effect on many graphs. As long as every vertex has moderate degree, connections between them quickly make certain vertices of high degree, which will in turn connect to and raise the degrees of other vertices, and so forth until every edge is in the graph. We can thus see that the closure procedure, in a good number of cases, will yield a complete graph, and, since K_n is Hamiltonian for $n \geq 3$, we could characterize a sufficient condition for Hamiltonicity by finding what degree conditions set off such an avalanche of edge-addition in the closure:

Theorem 2 (Chvátal, '72)**.** *Let G have $n \geq 3$ vertices of degrees $\delta(G) = d_1 \leq d_2 \leq \cdots \leq d_n = \Delta(G)$. If for all $i < \frac{n}{2}$, either $d_i > i$ or $d_{n-i} \geq n - i$, it follows that G is Hamiltonian.*

Proof. As mentioned above, we want to show that graphs satisfying the above criterion have K_n as their closure; the Bondy-Chvátal result would then guarantee Hamiltonicity. It is, however, easier to prove the contrapositive of this result: if a graph does *not* have a complete graph as its closure, then the above criterion is not satisfied. We shall in fact prove a slightly stronger statement: if a graph does not have a complete graph as its closure, then the above criterion is not satisfied on its closure, either. This is a stronger statement since the degrees in the closure will be larger than those in the graph itself, so if the closure does not satisfy the Chvátal criterion, neither does the underlying graph.

We shall thus consider a closed graph G on n vertices not equal to K_n, and show that it does not satisfy the Chvátal criterion. Since G is not a complete graph, it contains at least one pair of nonadjacent vertices. Let u and v be a pair of nonadjacent vertices with the largest degree sum among all pairs of nonadjacent vertices in the graph with $d(u) \leq d(v)$, and since G is closed, it must be the case that $d(u) + d(v) < n$, so one of $d(u)$ or $d(v)$ is less than $\frac{n}{2}$; without loss of generality let us say $d(u) < \frac{n}{2}$.

Since $d(u) + d(v)$ was chosen to be as large as possible for $u \not\sim v$, it is the case that for all $w \not\sim v$, $d(w) \leq d(u)$. We know there are $n - d(v) - 1$ non-neighbors of v; since $d(u) + d(v) < n$, it follows that $n - d(v) - 1 > d(u) - 1$, so v has at least $d(u)$ neighbors. Thus, there are $d(u)$ vertices of degree no more than $d(u)$.

Similarly, since $d(u) + d(v)$ was chosen to be as large as possible for $u \not\sim v$, it is the case that for all $w \not\sim u$, $d(w) \leq d(v) < n - d(u)$. We know there are $n - d(u) - 1$ non-neighbors of u, and in addition, u itself has degree no more than $d(v)$, so u's non-neighbors together with u itself form a set of $n - d(u)$ vertices of degree less than $n - d(u)$.

There are $d(u)$ vertices of degree less than or equal to $d(u)$, so placing the vertices in ascending order by degree, $d_{d(u)} \leq d(u)$; likewise, there are $n - d(u)$ vertices of degree less than $n - d(u)$, so $d_{n-d(u)} < n - d(u)$. Thus, for the specific value $i = d(u)$, G can be seen to violate the Chvátal criterion. □

Unfortunately, the Chvátal criterion isn't a lot of help in many cases: it can't detect large cycles with several edges added, since such a structure will probably have $d_2 = 2$, but is fairly unlikely to have $d_{n-2} \geq n - 2$.

There are few other simple sufficient conditions for Hamiltonicity of graphs, but there's one untapped possibility: connectivity, and particularly 2-connectivity, is associated with the presence of cycles. There is in fact a connectivity-related condition for Hamiltonicity, althoguh it surprisingly introduces another parameter:

Theorem 3 (Chvátal and Erdős, '72)**.** *For $|G| > 3$, if $\kappa(G) \geq \alpha(G)$, then G has a Hamiltonian cycle.*

Proof. If G is a complete graph, it is trivially Hamiltonian; for all other G, $\alpha(G) \geq 2$. Thus, let us consider G in which $\kappa(G) \geq \alpha(G) \geq 2$. Let C be a longest cycle in G.

We know that $\delta(G) \geq \kappa(G)$, and furthermore it is quite easy to show that G contains a cycle with at least $\delta(G) + 1$ vertices: a maximal path in G must contain every neighbor of its endpoints and thus have length at least $\delta(G)+1$; the edge between an endpoint and the furthest along this path of its neighbors produces a cycle of length at least $\delta(G)+1$, so C contains at least $\delta(G)+1 \geq \kappa(G)+1$ vertices.

If C is Hamiltonian, then we are obviously done; if it is not, then $G - V(C)$ is nonempty, so let us consider any component H of $G - V(C)$. We will know that the vertices of H have at least $\kappa(G)$ neighbors in C, since removing H's neighbors in C disconnects C from H, which would require $\kappa(G)$ vertex-removals. If C consists of vertices $v_1 \sim v_2 \sim \cdots \sim v_r \sim v_1$, let us denote the vertices of C adjacent to vertices of H by $v_{i_1}, v_{i_2}, \ldots, v_{i_{\kappa(G)}}$, with indices in ascending order.

For each i_j, we shall show that C can be extended (contradicting maximality) if either v_{i_j+1} is adjacent to some vertex of H, or v_{i_j+1} is adjacent to some v_{i_k+1}.

If v_{i_j+1} is adjacent to some vertex of H, then by connectedness of H there is a path from v_{i_j}'s neighbor in H to v_{i_j+1}'s neighbor in H. Splicing this path in to replace the edge $v_{i_j} \sim v_{i_j+1}$ will extend C.

On the other hand, if v_{i_j+1} is adjacent to some v_{i_k+1} (with $k > j$, for simplicity), then we may make use of a path through H from v_{i_j} to v_{i_k} to construct the longer cycle:

$$v_1 \sim \cdots v_{i_j} \sim H \sim v_{i_k} \sim v_{i_k-1} \sim \cdots \sim v_{i_j+1} \sim v_{i_k+1} \sim v_{i_k+2} \sim \cdots \sim v_r \sim v_1$$

Thus, the assertion that C is the longest cycle in the graph guarantees that each of the v_{i_j+1} are mutually nonadjacent and not adjacent to any vertices of H. Taken collectively, they form an independent set nonadjacent to H, so we may add any vertex of H to get an independent set of size $\kappa(G) + 1$, contradicting our condition $\alpha(G) \leq \kappa(G)$; since the possibility that $G - V(C)$ is nonempty leads to a contradiction, C is a Hamiltonian cycle. □

One interesting example of a Hamiltonian tour frequently encountered in recreational mathematics is the famous chess problem of the *Knight's Tour*: can a chess knight, a piece which moves 2 steps in one direction and then a third step perpendicular to its path, visit every square of a chess board and return to its starting point? Does its ability to do so depend on the dimensions of the board? The knight's mobility can be modeled as a graph: every square of the board can be identified with an vertex, with edges between vertices which are one step away from each other, then "visiting every square" translates to visiting every vertex, and the knight's tour is simply a matter of finding a Hamiltonian cycle in this graph. The bad news is: none of the techniques we learned here will actually help us much! The graph in question has 64 vertices (in the traditional 8×8 chessboard), with degrees ranging from 2 to 8, so closures aren't illuminating; its independence number is far larger than its connectivity, so the Chvátal-Erdős theorem is useless; nonetheless, it has Hamiltonian cycles, which were traditionally shown by exhaustive construction and have more recently been addressed with specialized recreational-mathematics theorems such as the below (not proved in this class):

Theorem 4 (Schwenk, '91)**.** *There is a knight's tour on an $m \times n$ board if and only if either $(m, n) = (1, 1)$ or all of the following conditions hold:*

- *m and n are not both odd,*
- *neither m nor n is 1, 2, or 4,*
- *the board is not 3×4, 3×6, or 3×8.*

Showing that these forbidden boards are in fact impossible to tour is pretty easy; showing that every other board is possible to tour is quite difficult.

We'll close off this topic by looking, very briefly, at a simpler sort of Hamiltonian structure.

1.2 Hamiltonian Paths

A Hamiltonian path is, of course, a path visiting every vertex of a graph, analogous to an Eulerian trail. The criterion for containing an Eulerian trail, we may recall, was almost as simple as the criterion for containing an Eulerian tour. We can certainly relate Hamiltonian paths to cycles, in ways which range from the vacuous to the slightly subtle:

Proposition 3. *If G is Hamiltonian, then G contains a Hamiltonian path. Furthermore, for any edge e or vertex v, $G - e$ and $G - v$ also contain a Hamiltonian path.*

Proof. Removal of an edge from a Hamiltonian cycle produces a path; likewise, removing a vertex from the graph will remove a vertex and two edges from the cycle, producing a Hamiltonian path on $G - v$. □

Proposition 4. *If G contains a Hamiltonian path, then there are vertices u and v such that for $e = \{u, v\}$, $G + e$ is Hamiltonian.*

Proof. Let $v_1 \sim v_2 \sim \cdots \sim v_n$ be the Hamiltonian path; clearly for $e = \{v_1, v_n\}$, $G + e$ contains the Hamiltonian cycle $v_1 \sim v_2 \sim \cdots \sim v_n \sim v_1$. □

There is one interesting (and pelasingly bidirectional) result putting to rest all questions involving Hamiltonian paths, or at least phrasing them in terms of the more well-researched if not actually easier question of Hamiltonian cycles.

Proposition 5. *Let the* cone $C(G)$ *be the graph produced from* G *by adding a vertex* x *adjacent to every vertex in* G*.* G *has a Hamiltonian path if and only if* $C(G)$ *is Hamiltonian.*

Proof. If G contains a Hamiltonian path $v_1 \sim v_2 \sim \cdots \sim v_n$, then $C(G)$ contains the Hamiltonian cycle $v_1 \sim v_2 \sim \cdots \sim v_n \sim x \sim v_1$, so $C(G)$ is Hamiltonian.

Conversely, if $C(G)$ is Hamiltonian, it contains a cycle, which we shall orient with x first so that it can be expressed as $x \sim v_1 \sim v_2 \sim \cdots \sim v_n \sim x$. Then $v_1 \sim v_2 \sim \cdots \sim v_n$ is a Hamiltonian path in G. □