# 1 Advanced Chromatic Properties

## 1.1 Color-Criticality

When we were proving particular results in chromatic number, such as the five-color theorem, we frequently assumed we were looking at a *minimal* example of some species of graph. Very often, the minimal examples of a problem have exploitable structures we can make better use of. Such graphs are called *critical*:

**Definition 1.** A graph $G$ without isolated vertices is *color-critical* if $\chi(G - e) < \chi(G)$ for any $e \in E(G)$. If $\chi(G) = k$, this property may be further described as *$k$-criticality.*

We know a few color-critical graphs off the top of our heads: $K_n$ will be $n$-critical, since removal of any edge allows coloring with $n - 1$ colors, and $C_{2n+1}$ will be 3-critical, since removing any edge admits a 2-coloring. In fact, every graph has a $k$-critical "core", which in many cases is not a cycle or a clique.

**Proposition 1.** *If $\chi(G) = k$, then $G$ has a $k$-critical subgraph.*

*Proof.* Let us prove this by induction on $\|G\|$; for the base case, note that $\|G\| = 1$ corresponds uniquely to a 2-critical graph.

For larger $G$, one of two things is true: either $G$ is $k$-critical, in which case it is its own $k$-critical subgraph, or there is an edge $e$ such that $\chi(G - e) \geq \chi(G)$. Since any coloring of $G - e$ is a proper coloring of $G$, we know that this nonstrict inequality is in fact an equality; that is, $\chi(G - e) = \chi(G) = k$. Then, by the inductive hypothesis, since $\|G - e\| < \|G\|$ and $\chi(G - e) = k$, $G - e$ has a $k$-critical subgraph, which is in turn a $k$-critical subgraph of $G$. □

Since we now know $k$-critical graphs are all over the place, we might begin to wonder about their structure, since they are exemplars of the necessary substructures for a graph to require $k$ colors.

**Proposition 2.** *If $G$ is $k$-critical, then $\delta(G) \geq k - 1$.*

*Proof.* Suppose $\chi(G) = k$ and $G$ has a vertex $v$ with degree less than $k - 1$. Let $e = \{u, v\}$ for some neighbor $u$ of $v$. If $\chi(G - e) = k - 1$, then it must follow that in this coloring $u$ and $v$ are the same color (or this would be a $(k - 1)$-coloring of $G$ as well; however, since $v$ has no more than $k - 3$ neighbors in $G - e$, there must be two colors $\leq k - 1$ not represented in its neighborhood, so any proper coloring of $G - e$ except for $v$ allows at least two choices of color for $v$, so it can always be selected to be a different color than $u$, inducing a $(k - 1)$-coloring of $G$, which contradicts the given chromatic number of $G$. Thus, $\chi(G - e)$ must be $k$, so $G$ is not $k$-critical. □

Not only is the minimum degree at each vertex necessarily close to the chromatic number of a critical graph, but in fact, the more global concept of edge-connectivity must also be dictated by the chromatic number. We might start with a quite simple observation.

**Proposition 3.** *Every $k$-critical graph is connected.*

*Proof.* Suppose a $k$-chromatic graph $G$ had two or more components $G_1, \ldots, G_r$. If any of these components are isolated vertices, $G$ is definitionally non-$k$-critical. Otherwise, each of these components contains at least one edge. We know that $\chi(G) = \max(\chi(G_1), \ldots, \chi(G_r))$, so since

$\chi(G) = k$, some $\chi(G_i) = k$; without loss of generality let us declare $\chi(G_1) = k$. Then, if $e \in G_2$, $\chi(G - e) = \max(\chi(G_1), \chi(G_2 - e), \chi(G_3), \ldots, \chi(G_r)) = k$, so $G$ is not $k$-critical. □

We then show that if a graph is partitioned into two induced subgraphs of smaller chromatic number, there must be several edges between the two parts:

**Lemma 1** (Dirac '53). *If $\chi(G) > k$, and the vertices in a $G$ are partitioned into two sets $A$ and $B$, with the induced graphs $G[A]$ and $G[B]$ being $k$-colorable, there must be at least $k$ edges between the two sets.*

*Proof.* Let us consider a specific $k$-coloring of $A$ and $B$, and partition $A$ and $B$ into color-classes $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$. Since we know $G$ as a whole is *not* $k$-colorable, we expect the edges between $A$ and $B$ to be such that they impede joining the colorations of $A$ and $B$ seamlessly, even if we permute the colors on one or the other of them.

To this end we shall form a bipartite graph $H$ with vertices $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$, such that $u_i \sim v_j$ if there is no edge in $G$ between the sets $A_i$ and $B_j$. A perfect matching on this graph would provide a permutation of colors on $B$ so as to allow the $k$-colorings of $A$ and $B$ to be merged, since a perfect matching on $H$ corresponds to $k$ disjoint pairs of sets $(A_i, B_j)$ between which there are no edges, and which can thus be put into the same color-class with impunity.

Since $G$ is not $k$-colorable, we thus know that $H$ must not have a perfect matching. On this basis we want to show that there are several edges not in $H$: recall our original goal of finding edges between $A$ and $B$. By construction of $H$, every absent edge in $H$ corresponds to at least one edge in $G$, so our goal is to show that $\|H\| \leq \|K_{k,k}\| - k = k^2 - k$. This is actually fairly easy, and could even be shown directly, but doing so is a bit messy; an elegant and straightforward way to show that any subgraph of $K_{k,k}$ with more than $k^2 - k$ edges has a perfect matching is to invoke König-Egerváry; recall that a bipartite graph would have a maximal matching on $k$ edges of the same size as its smallest vertex-cover. However, such a graph has maximum degree $k$, so every vertex covers at most $k$ edges, so if there are more than $k(k-1)$ edges in the graph, the smallest vertex cover has size at least $k$, and thus the maximal matching has size at least $k$, which, given that there are $k$ vertices in each part, necessitates a perfect matching.

Thus, since $H$ has no perfect matching, it has $k^2 - k$ or fewer edges, and thus there are at least $k$ nonadjacent pairs $(u_i, v_j)$, which correspond to sets $(A_i, B_j)$ between which there is at least one edge. Since there are at least $k$ such pairs, there are at least $k$ edges from $A$ to $B$. □

This is not quite a statement about criticality, though, although like statements about criticality it involves a concept of reducing a graph's size necessitating a reduction in chromatic number. This similarity, however, will give us the connectivity criterion we seek.

**Theorem 1** (Dirac '53). *If $G$ is $k$-critical, then it is $(k-1)$-edge-connected.*

*Proof.* Let $S$ be a set of edges in $G$ such that $G - S$ is disconnected. Let $A$ and $B$ be a partition of the vertices in $G$ such that there is no edge between $A$ and $B$ in $G - S$ (if $G - S$ has two components, $A$ and $B$ will each be one component; if $G - S$ has more components, $A$ and $B$ can be any partition of the components in which neither part is empty). The induced graphs of $G$ on the vertex-sets $A$ and $B$ are smaller than $G$; since $G$ is $k$-critical, it must thus be the case that these induced graphs have chromatic number less than $k$. This partition $(A, B)$ thus satisfies the conditions of the previous lemma, considering $k - 1$ to take the place of every occurrence of $k$. Since there are no edges between $A$ and $B$ in $G - S$, every edge between $A$ and $B$ must lie in $S$; by the above lemma, we know there are at least $k-1$ such, so $|S| \geq k-1$; since a separating set of edges must be of size $k-1$ or larger, we know $G$ is $(k-1)$-edge-connected. □

On the subject of vertex-connectivity of $k$-critical graphs we cannot say nearly as much. It's not hard to see that every connected $k$-critical graph with $k \geq 3$ is 2-connected; if a $k$-critical graph $G$ had a cut-vertex $v$, then $G - v$ would be $(k-1)$-colorable by criticality, and specifically the individual components $G_1$ and $G_2$ thereof would be $(k-1)$-colorable. Since $G - e$ is also $(k-1)$-colorable for every edge $e$ incident on $v$, note that if we consider separately edges from $v$ to $G_1$ and $G_2$, we see that $v \cup G_1$ and $v \cup G_2$ have $(k-1)$-colorings, which can be reconciled by an appropriate permutation of the colors to be equal on $G$, giving a $(k-1)$-coloring of $G$.

However, 2-connectivity is the best we can do, in general:

**Proposition 4.** *A graph $G$ with vertices $u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1}, w$ and edges $u_i \sim u_j$ and $v_i \sim v_j$ for all $i$, $j$, as well as $u_i \sim w$ and $v_i \sim w$ for $i \neq 1$, and lastly $u_1 \sim v_1$, is 2-connected and $k$-regular.*

*Proof.* First, it is easy to show this graph is 2-connected: the only paths from the $u_i$ vertices to the $v_i$ vertices are through the edge $u_1 \sim v_1$ and the vertex $w$, so both $\{u_1, w\}$ and $\{v_1, w\}$ are vertex cutsets.

Now, to show $k$-criticality, we start by showing this graph is $k$-chromatic. Let us note that the $u_i$ vertices and $v_i$ vertices, respectively, are $K_{k-1}$ subgraphs, and thus each use $k-1$ different colors. $u_1$ and $v_1$ must be different colors since they are adjacent, so if we attempt to color with only $k-1$ colors, the color present on $u_1$ must appear on some $v_i$ with $i \neq 1$. Since $w$ is adjacent to $v_i$ as well as $u_1, \ldots, u_{k-1}$, we see that $w$ is adjacent to $k-1$ different colors, and thus does ont admit a proper $(k-1)$-coloring, so $G$ is not $(k-1)$-colorable. On the other hand, it is $k$-colorable, since we may assign each $u_i$ the color $i$, $v_i$ the color $k-i$, and $w$ the color $k$.

Now, we shall see that removing any edge from this graph makes it $(k-1)$-chromatic. There are several possible edges: we could remove $\{u_1, v_1\}$, $\{u_1, u_i\}$ for some $i$, $\{u_i, u_j\}$ for some $i, j \neq 1$, $\{u_i, w\}$ for some $i \neq 1$, $\{v_1, v_i\}$ for some $i$, $\{v_i, v_j\}$ for some $i, j \neq 1$, or $\{v_i, w\}$ for some $i \neq 1$. This is a long list, but we can cut it nearly in half by appealing to symmetry: any argument to be made about $u_i$ vertices also applies to $v_i$ vertices, so there are only 4 actual cases.

One of the easiest cases is to color $G - \{u_1, v_1\}$: we can color each $u_i$ and $v_i$ with color $i$, and use color 1 on $w$. If we are coloring $G - \{u_1, u_i\}$ or $G - \{u_i, u_j\}$, by $(k-1)$-criticality of $K_{k-1}$, we can color all the $i$ vertices with $k-2$ colors; let us do so using color 1 at $u_1$. Then we color each $v_i$ with $k-i$, and $w$ with color $k-1$. Finally, if we are coloring $G - \{u_i, w\}$, let us color $u_i$ in color 1, and the remaining $u$ vertices in colors $2, \ldots, k-1$, and then color $v_i$ with color $i$ and $w$ in color 1.

Thus, $G - e$ is $(k-1)$-colorable for all $e$, so $G$ is $k$-critical. □

The above is a minimal example of a more general $k$-critical construction of Hajós.

**Proposition 5** (Hajós, '61)**.** *If $G_1$ and $G_2$ are $k$-critical, the following construction $G_1 \oplus G_2$ is both $k$-critical and 2-connected: for $\{u_1, v_1\} \in E(G_1)$ and $\{u_2, v_2\} \in E(G_2)$, let $V(G_1 \oplus G_2) = V(G_1) \cup V(G_2) \cup \{v\} - \{v_1, v_2\}$, and then let edge adjacencies be as such: $x \sim y$ if $x \sim y$ in $G_1$ or $G_2$; $x \sim v$ if $u_1 \neq x \sim v_1$ or $u_2 \neq x \sim v_2$ in $G_1$ or $G_2$ respectively; and lastly, $u_1 \sim u_2$. In other words, $G_1 \oplus G_2$ is produced by taking a union of the two graphs, identifying two vertices $v_1$ and $v_2$ with each other, removing edges from the new vertex $v$ to specific neighbors $u_1$ and $u_2$, and adding the edge $\{u_1, u_2\}$.*

*Proof.* The proof of this is actually quite similar to the special case: we show that $\chi(G_1 \oplus G_2) = k$, and then we consider the following possible edges to be removed from $G_1 \oplus G_2$: the new edge $\{u_1, u_2\}$, an edge from $u_1$ to elsewhere in $G_1$, an edge internal to $G_1$, an edge from $G_1$ to $v$, an edge from $u_2$ to elsewhere in $G_2$, an edge internal to $G_2$, or an edge from $G_2$ to $v$. Symmetry of construction allows us to dispense with the last 3 cases, and the first 4 can be managed by an invocation of $k$-criticality of $G_1$ and $G_2$ not unlike that used in the special case above. □

## 1.2 Perfect Graphs

Another situation of interest in advanced coloring explorations is the relationship between the coloring number and the clique number: we know that $\chi(G) \geq \omega(G)$, but that there are many graphs for which equality does not hold, the smallest such being $C_5$. However, we would expect that for "most" graphs, $chi(G)$ will equal $\omega(G)$; after all, many graphs contain large cliques. The situation becomes interesting, however, when we demand not only that $G$ but that all of its substructures have equal chromatic and clique nubmers, which takes the equality from a "local" property (presence of a large clique somewhere) to a "global" one (every substructure of the graph has an appropriately large clique.

**Proposition 6.** *A graph $G$ is* perfect *if, for every induced subgraph $H$ of $G$, $\chi(H) = \omega(H)$.*

So this is much more restrictive than mere equality of $\chi(G)$ and $\omega(G)$; if $G$ contained *any* large odd induced cycles, then $G$ would not be perfect. This intuition that any large cycles in $G$ must not be induced cycles gives a quantifiable definition of that concept.

**Definition 2.** An edge $e$ is a *chord* of a cycle in a graph $G$ if its endpoints are nonconsecutive points of the cycle. A graph $G$ is *chordal* if every cycle of 4 or more vertices in $G$ contains a chord.

These graphs are promising as pertains to perfection, since they cannot contain any induced cycle larger than a $C_3$; any larger cycle in the graph has a chord and thus would not be simply cycles in the induced subgraph on the vertices. Every chordal graph is in fact perfect, but we'll end up needing a chordal structural theorem to show this to be so.

**Definition 3.** A graph $G$ is the result of *gluing* two graphs $G_1$ and $G_2$ along some set of vertices $S$ if there are edge-preserving maps from $G_1$ and $G_2$ to $G$ which cover all vertices and edges of $G$ and overlap on $S$; alternatively, a graph is the result of gluing $G_1$ and $G_2$ along $S$ if there are subsets $V_1$ and $V_2$ of $V(G)$ such that $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = S$, and $G_1$ and $G_2$ are respectively isomorphic to the induced subgraphs $G[V_1]$ and $G[V_2]$, and every edge lies in $G[V_1]$ or $G[V_2]$.

**Theorem 2** (Structural Theorem for Chordal Graphs)**.** *Every graph is chordal if and only if it is either a clique or the result of gluing smaller chordal graphs together along cliques.*

*Proof.* Trivially the clique itself is chordal, so we shall start by proving that clique-gluings of chordal graphs are chordal. Consider a cycle in a graph $G$ resulting from gluing chordal $G_1$ and $G_2$ along a clique $S$. If all vertices of the cycle lie in $G_1$ or $G_2$, then by chordality of $G_1$ or $G_2$ the cycle has a chord. If the cycle has vertices from both $G_1 - G_2$ and $G_2 - G_1$, then since $S$ separates these two sets, the cycle must have two vertices from $S$ nonadjacent in the cycle, which will agian be a chord, since $S$ is a clique and has an edge between any two vertices. Thus, any cycle in $G$ has a chord, so $G$ is chordal.

Conversely, suppose $G$ is a chordal graph; if $G$ is a clique it clearly satisfies this description, so let us suppose it is not a clique. Consider non-adjacent $u$ and $v$ in $G$. Let us consider $S$ given as a minimal set of vertices separating $u$ and $v$, so $G - S$ can be divided into disconnected parts containing $u$ and $v$ respectively. Let $G_1$ consist of the part containing $u$ and $S$; let $G_2$ be the induced graph on the remainder of $G$ together with $S$. By construction $G$ is a gluing of $G_1$ and $G_2$ along $S$; we need to show that $G_1$ and $G_2$ are chordal, and that $S$ is a clique. The first of these is easy: any cycle in $G_1$ or $G_2$ is a cycle in $G$, and thus has a chord; the chord must then also lie in $G_1$ or $G_2$, since they are induced subgraphs of $G$. To show $S$ is a clique, let us suppose to the contrary that there are $s_1, s_2 \in S$ which are non-adjacent. Since $S$ is a minimal separating set, both $s_1$ and $s_2$ have neighbors in $G_1$, so there are paths from $s_1$ to $s_2$ in $G_1$; choose the shortest such path, and likewise, choose the shortest such path in $G_2$. These paths together form a cycle of length at least 4, but cannot have any chords, since any chords would either connect $G_1$ and $G_2$, or shorten the paths described. But $G$ is chordal, so this structure cannot exist, so $s_1$ and $s_2$ are adjacent. □

From this structural theorem, it ends up being fairly trivial to prove that chordal graphs are perfect:

**Proposition 7.** *If $G$ is chordal, then $G$ is perfect.*

*Proof.* Cliques are known to be perfect; the structural theorem says that chordal graphs are formed from gluing together smaller chordal graphs, repeating the gluing procedure at each level until we get down to cliques; thus, all that needs to be proven is that a gluing along a clique of two perfect graphs is perfect. Let $G$ be the gluing of perfect $G_1$ and $G_2$ along $S$, as usual, and consider an induced subgraph $H$ of $G$. If $H$ lies entirely in $G_1$ or $G_2$, then $\chi(H) = \omega(H)$ by perfection of the containing graph, so let us consider the case where $H$ has nontrivial sections $H_1 = H \cap G_1$, $H_2 = H \cap G_2$, and $H' = H \cap S$. Since $H_1$ and $H_2$ are induced subgraphs of perfect graphs $G_1$ and $G_2$, we know $\chi(H_1) = \omega(H_1)$ and $\chi(H_2) = \omega(H_2)$. Finally, $H'$ is an induuced subgraph of clique $S$, so $H'$ is a clique. We thus know that on $H' = H_1 \cap H_2$, every color is used only once, so it is necessarily possible to permute $H_1$'s colros to match those of $H_2$, and reconcile the colorings, so $\chi(H) \leq \max(\chi(H_1), \chi(H_2) \leq \max(\omega(H_1), \omega(H_2)) = \omega(H)$, so $\chi(H) \leq \omega(H)$, which must be equality since the chromatic nubmer is always at least the clique number. □

There is a specific popular family of graphs that is easy to show is chordal, and thus perfect.

**Definition 4.** An *interval graph* is a graph $G$ whose vertices are open intervals of $\mathbb{R}$, and in which vertices are adjacent if they overlap.

**Proposition 8.** *Every interval graph is chordal, and thus perfect.*

*Proof.* Let an interval graph $G$ contain a cycle $v_1 \sim v_2 \sim v_3 \sim \ldots \sim v_k \sim v_1$ with $k \geq 4$, with $v_i$ associated with the interval $[a_i, b_i]$. Suppose this cycle has no chord. Let us consider the 90 different orderings of $a_{i-1} < b_{i-1}, a_i < b_i, a_{i+1} < b_{i+1}$ (in fact, we only really need to consider fifteen cases, if we ignore order). There are only two orderings in which $v_{i-1} \sim v_i \sim v_{i+1}$ and $v_{i-1} \not\sim v_{i+1}$:

$$a_{i-1} < a_i < b_{i-1} < a_{i+1} < b_i < b_{i+1}$$

$$a_{i+1} < a_i < b_{i+1} < a_{i-1} < b_i < b_{i-1}$$

Thus, for all $i$, it is the case that either $a_{i-1} < a_i < a_{i+1}$ or $a_{i+1} < a_i < a_{i-1}$. In addition, the *same* inequality must be true for all $i$, since the boundary between a strictly-increasing and strictly-decreasing sequence would consist of three values which are not monotonic.

Without loss of generality, we may thus say that

$$a_1 < a_2 < b_1 < a < 3 < a_4 < a_5 < \cdots < a_k$$

But then $v_1 \not\sim v_k$, violating our presumption that this was a cycle. □

Individual collections of perfect graphs do little to illuminate the concept of perfection, however. While chordal graphs are a good example of perfection, there are clearly perfect graphs that are not chordal, such as even cycles.

In seeking out universal properties of chordal graphs, Berge noted that a graph's perfection seemed intimitely related to the perfection of its complement. This is of some interest since $\omega(G^c)$ is a fundamental graph property, the independence number $\alpha(G)$. Less obviously, $\chi(G^c)$ also has a straightforward interpretation: it is the minimum number of disjoint cliques needed to cover every vertex of $G$, since if $G^c$ is $r$-colorable, then each of the $r$ color classes in $G^c$ is a clique in $G$.

In one case, the property Berge noted turns out to rely fundamentally on other, well-known graph concepts:

**Proposition 9** (Gallai, '58)**.** *If $G$ is bipartite (and thus perfect), then $G^c$ is perfect.*

*Proof.* Since any induced subgraph of a bipartite complement is itself a bipartite complement, it will suffice to show that $\omega(G^c) = \chi(G^c)$. Using the characterization of $\omega(G^c)$ and $\chi(G^c)$ above, we note that these quantites are respectively the maximum number of independent vertices in $G$ and the minimum number of cliques which cover every vertex of $G$. We may assume $G$ has no isolated vertices; any such vertices will clearly contribute exactly one to both $\omega(G^c)$ and $\chi(G^c)$, and would have no effect on perfection. Interpreting $\chi(G^c)$ as equal to the minimum number of disjoint cliques which cover all the vertices $G$, we can note that since $\omega(G) = 2$, all such cliques will be either edges or single vertices; using as many edges as possible will yield the least cover, so for $M$ a maximal matching on $G$, we see that there are $|G| - 2\|M\|$ uncovered vertices left, so $\chi(G^c) = \|M\| + (|G| - 2\|M\|) = |G| - \|M\|$. We shall show that $\omega(G^c) \geq \chi(G^c)$ (with equality guaranteed since clique number cannot exceed chromatic number) by explicitly constructing an independent set of $|G| - \|M\|$ elements. By the König-Egerváry Theorem, there is a set $S$ of vertices with $|S| = \|M\|$, such that every edge of $G$ is incident on a vertex of $S$. Then, $V(G) - S$ must be an independent set, since if any two vertices not in $S$ were adjacent, the edge between them would not be covered by $S$. □

Berge's full observation, known as the *weak perfect graph conjecture* went unproven for several years, but was finally definatively shown by Lovász.

**Lemma 2** (Lovász '72)**.** *If $G$ is perfect with $v \in V(G)$, and $G' = G + v'$, where $v'$ is adjacent to every neighbor of $v$ and to $v$ itself, then $G'$ is perfect.*

*Proof.* We shall prove this by induction on $|G|$; $|G| = 1$ corresponds to the case $G = K_1$ and $G' = K_2$, both of which are indeed perfect.

For our induction step, consider a graph $G$ and let $G'$ be the result of performing the "vertex-cloning" procedure above. For most induced subgraphs $H$ of $G'$, we can see that $\chi(H) = \omega(H)$ easily: if both $v$ and $v'$ aren't in $H$, then $H$ is isomorphic to a subgraph of $G$, and thus $\chi(H) = \omega(H)$ by perfection of $G$; if $v$ and $v'$ are in $H$ and $H$ is a proper subgraph of $G$, then consider the graph

$H^-$ produced by "uncloning" the $(v, v')$ pair; this is an induced subgraph of $G$, and thus is perfect, so by the induction hypothesis, $H^{-'}$ is perfect, but $H^{-'}$ is just $H$.

Thus, all we really need to do is show that $\chi(G') = \omega(G')$. For brevity, let us denote $\chi(G) = \omega(G)$ by $k$. We know that $\omega(G')$ is equal to either $k$ or $k+1$ by construction; likewise, $\chi(G')$ is either $k$ or $k+1$, since we're adding a single vertex which, at worst, would require one new color. The only case here which would be problematic is if $\chi(G') = k+1$ and $\omega(G') = k$. Thus, we know that $v$ is not in a maximal clique of $G$, or the introduction of $v'$ would increase the size of that clique. Consider a $k$-coloring of $G$ such that $v$ has color $k$; note that color $k$ appears both at $v$ and in every $K_k$ of $G$; let $S$ consist of every vertex of color $k$ except $v$, and let us consider the induced graph $H$ of $G$ on $V(G) - S$. By the above observation, we know $\omega(H) < k$, so $\chi(H) < k$. A $(k-1)$-coloring of $H$ is very nearly a coloring of $G'$, but it has left all vertices of $S$, as well as $v'$, uncolored. But by construction the elements of $S$ are mutually nonadjacent, and since they were all nonadjacent to $v$, they're non-adjacent to $v'$ as well, so the $(k-1)$-coloring of $H$ can be easily extended to a $k$-coloring of $G'$ by coloring $S$ and $v'$ in color $k$; thus, if $\omega(G') = k$, $\chi(G') = k$. □

**Theorem 3** (Perfect Graph Theorem, Lovász '72)**.** *If a graph $G$ is perfect, so is $G^c$.*

*Proof.* We prove this by induction on $|G|$. $|G| = 1$ is a trivial case.

We now want to show that every induced subgraph $H$ of $G$ has $\chi(H^c) = \omega(H^c)$. Since every induced subset $H$ of $G$ is perfect, we can invoke the inductive hypothesis on any *proper* induced subgraph $H$ to see that $H^c$ is perfect and thus $\chi(H^c) = \omega(H^c)$. Thus, the only case we actually need to prove is specifically that $\chi(G^c) = \omega(G^c)$.

Let $\mathcal{S}$ be the set consisting of all sets of vertices which form cliques. Let $\mathcal{A}$ be the set consisting of all *maximal* independent sets in $G$, so that every element of $\mathcal{A}$ has size $\alpha(G)$. We shall now find a clique $S$ in $G$ which intersects every single element of $\mathcal{A}$; if this can be done, we know that $\omega(G^c - S) = \alpha(G - S) < \alpha(G) = \omega(G^c)$, and since $S$ is a clique, and can thus be all a single color in $G^c$, $\chi(G^c) \leq \chi(G^c - S) + 1$. Putting these two inequalities together and invoking the induction hypothesis on $G - S$:

$$\chi(G^c) \leq \chi(G^c - S) + 1 = \omega(G^c - S) + 1 \leq \omega(G^c)$$

which we know to be equality since the clique number is no larger than the chromatic number.

However, we must prove the existence of such a clique $S$. Suppose no such clique exists. Then, for every clique $S$, there is at least one element $A(S)$ of $\mathcal{A}$ not intersecting it. Let $k(v)$ be equal to the number of cliques $S$ such that $v \in A(S)$, and now we shall construct a graph $G'$ by replacing every vertex $v$ with a clique of size $k(v)$, preseving adjacencies on each vertex. This could also be expressed as the result of a great many of the above-described "vertex-clone" procedures, so by the lemma we know that $G'$ is perfect, and that $\chi(G') = \omega(G')$. We shall see that this is not actually possible by explicitly determining $\chi(G')$ and $\omega(G')$.

We know any clique in $G$ will be a clique in $G'$ after inflating each vertex, albeit a bigger clique. We thus know that for some clique $S$ in $G$, $\omega(G') = \sum_{v \in S} k(v)$, which is by the definition of $v$ equal to the number of pairs of vertices $v$ from $S$ and cliques $T$ such that $v \in A(T)$. Ranging over all values of $v$, we see that for any given $T$, the number of $v \in A(T)$ is $|S \cap A(T)|$, so $\omega(G') = \sum_T |S \cap A(T)|$; since $S$ is a clique and $A(T)$ an independent set, we know the summand here is at most 1, and is zero at least once, when $T = S$, so $\omega(G') < |\mathcal{S}|$.

But in contrast, let us inspect $\chi(G')$. We'll start out by discussing independence number: we know that at each vertex we inflated $G$ by a factor of $k(v)$, so $|G'| = \sum_{v \in G} k(v)$, and using the same approach which was in the last paragraph limited to $S$, we see that $|G'|$ is equal to the number of pairs of vertices $v$ in $G$ and cliques $T$ such that $v \in A(T)$, and as above we shall see that for any given $T$ there are $|A(T)|$ such pairs, and thus $|G'| = \sum_T |A(T)| = \sum_T \alpha(G) = |\mathcal{S}|\alpha(G)$

Now, we will use this result in a chromatic bound we developed several weeks ago. Vertex-cloning doesn't increase independence number (since $v$ and $v'$ are dependent, and have the same neighborhoods), so $\alpha(G') = \alpha(G)$. Thus, $\chi(G') \geq \frac{|G'|}{\alpha(G')} = \frac{|\mathcal{S}|\alpha(G)}{\alpha(G)} = |\mathcal{S}|$.

Thus $\chi(G') \geq |\mathcal{S}| > \omega(G')$, contradicting perfection of $G'$. □

This is a fine characterization, but it is now, in fact, obsolete, thanks to a stronger conjecture of Berge, which was proven quite recently:

**Theorem 4** (Chudnovsky and Seymour, '02). *$G$ is perfect if and only if neither $G$ nor $G'$ contains an induced odd cycle on 5 or more vertices.*

The entirety of Chudnovsky and Seymour's proof is far too long to be shared here; it is based on an extremely complicated structural theorem on odd-cycle-free graphs, and then a demonstration that each step of the structural construction preserves perfection.