1. **(8 points)** *Evaluate the following limits; when a limit can not be evaluated, explicitly say so.*

   (a) **(2 points)** $\lim_{t\to -1} 3^t$.

   Since an exponential function with positive base is continuous throughout, we may use straightforward substitution to see that $\lim_{t\to -1} 3^t = 3^{-1} = \frac{1}{3}$.

   (b) **(2 points)** $\lim_{x\to 5} \frac{x^2-3x-10}{x^2-10x+25}$.

   Direct substitution yields the incalculable form $\frac{0}{0}$, and we shall thus try to factor out $(x-5)$ from the numerator and denominator. Since we look near, but not at, $x=5$, we can justify the straightforward cancellation arising therefrom:

$$\lim_{x\to 5} \frac{x^2-3x-10}{x^2-10x+25} \lim_{x\to 3} \frac{(x-5)(x+2)}{(x-5)(x-5)} = \lim_{x\to 3} \frac{x+2}{x-5}$$

   However, direct substitution in this new limit still yields an incalculable form, which is now $\frac{7}{0}$; since this has a nonzero numerator, it will correspond to some sort of infinite discontinuity, at which this limit does not exist.

   (c) **(2 points)** $\lim_{r\to 3} \frac{r^3-1}{r^2+r}$.

   Direct substitution yields a nonzero denominator, so $\lim_{r\to 3} \frac{r^3-1}{r^2+r} = \frac{3^3-1}{3^2+3} = \frac{26}{12} = \frac{13}{6}$.

   (d) **(2 points)** $\lim_{s\to -\infty} \frac{3s^3-2s^2+200}{3-7s^3}$.

   In the long term this function is dominated by its highest-degree terms in the numerator and denominator, so $\lim_{s\to-\infty} \frac{3s^3-2s^2+200}{3-7s^3} = \lim_{s\to-\infty} \frac{3s^3}{-7s^3} = \frac{-3}{7}$. Alternatively, one may formally divide by the highest degree appearing in the denominator, although doing so is a bit more involved:

$$\lim_{s\to-\infty} \frac{3s^3-2s^2+200}{3-7s^3} = \lim_{s\to-\infty} \frac{\frac{3s^3-2s^2+200}{s^3}}{\frac{3-7s^3}{s^3}} = \lim_{s\to-\infty} \frac{3s-\frac{2}{s}+\frac{200}{s^3}}{\frac{3}{s^3}-7} = \frac{3-0+0}{0-7} = \frac{-3}{7}$$

2. **(6 points)** *Let* $f(x) = \begin{cases} x^2 & \text{if } x<5 \\ 50-ax & \text{if } 5\le x<8 \\ \sqrt{x+b} & \text{if } x\ge 8 \end{cases}$.

   What choices of a and b will make this function continuous?

   Each of the individual parts of this function can be easily observed to be continuous on its domain, so problems can only arise at the junction points $x=5$ and $x=8$. To guarantee continuity at these points, we need to make sure that the left and right limits coincide, as such at $x=5$:

$$\begin{aligned} \lim_{x\to 5-} f(x) &= \lim_{x\to 5+} f(x) \\ \lim_{x\to 5-} x^2 &= \lim_{x\to 1+} 50-ax \\ 25 &= 50-5a \\ 5a &= 50-25 \\ a &= 5 \end{aligned}$$

And likewise for $x = 8$:

$$\begin{aligned}
\lim_{x\to 8-} f(x) &= \lim_{x\to 8+} f(x) \\
\lim_{x\to 8-} 50 - ax &= \lim_{x\to 8+} \sqrt{x+b} \\
50 - 5\cdot 8 &= \sqrt{8+b} \\
10 &= \sqrt{8+b} \\
100 &= 8+b \\
92 &= b
\end{aligned}$$

So our solution is to choose $a = 5$ and $b = 92$.

3. **(8 points)** *Determine the domains of the following functions:*

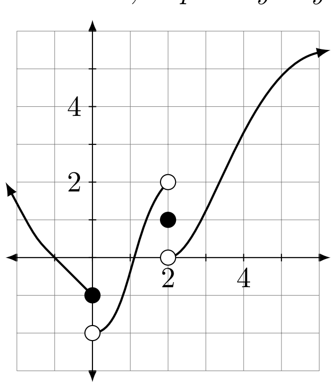
   (a) **(2 points)** $f(x) = \ln(7-2x)$.
   This function is evaluatable as long as the argument of the logarithm is positive; thus, the function's domain is the set of values where $7 - 2x > 0$; in other words, when $x < \frac{7}{2}$. In interval form, this would be $(-\infty, \frac{7}{2})$.

   (b) **(2 points)** $g(t) = \frac{\sqrt{t+6}}{t^2-9}$.
   This function is evaluatable as long as the argument of the square root is non-negative and the denominator of the fraction is nonzero; thus, the function's domain is the set of values where $t + 6 \geq 0$ and $t^2 - 9 \neq 0$; in other words, when $t \geq -6$ and $t \neq \pm 3$. In interval form, this would be $[-6, -3) \cup (-3, 3) \cup (3, \infty)$.

   (c) **(2 points)** $h(r) = \sqrt{r^2+9}$.
   This function is evaluatable as long as the argument of the square root is non-negative; note, however, that $r^2 + 9$ is always positive (and always greater than 9, at that). Thus, this function is evaluatable for all values of $\mathbb{R}$; in interval form, this would be $(-\infty, \infty)$.

   (d) **(2 points)** $q(t) = \sqrt{t} + \frac{7t}{3-2t}$.
   This function is evaluatable as long as the argument of the square root is non-negative and the denominator of the fraction is nonzero; thus, the function's domain is the set of values where $t \geq 0$ and $3 - 2t \neq 0$; in other words, when $t \geq 0$ and $t \neq \frac{3}{2}$. In interval form, this would be $[0, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$.

4. **(7 points)** *For the plot of $g(x)$ shown below, indicate whether or not each of the following quantities can be evaluated. If they can be evaluated, compute their values. If they cannot be evaluated, explicitly say so.*



$\lim_{x\to 0^-} g(x) = -1$, since immediately to the left of 0, the graph is very close to the height $y = -1$.

$\lim_{x\to 0^+} g(x) = -2$, since immediately to the right of 0, the graph is very close to the height $y = -2$.

$g(0) = -1$; note the solid dot at $(0, -1)$. $\lim_{x\to 2^-} g(x) = 2$, since immediately to the left of 2, the graph is very close to the height $y = 2$.

$\lim_{x\to 2^+} g(x) = 2$, since immediately to the right of 2, the graph is very close to the height $y = 0$.

$\lim_{x\to 2} g(x)$ does not exist, since the one-sided limits on the left and right, as seen above, do not agree.

$g(2) = 1$; note the solid dot at $(2, 1)$.

5. **(4 points)** *Given the function* $f(x) = \frac{4x^2-16}{x-3}$, *answer the following questions preparatory to sketching the functions.*

   (a) **(2 points)** *What is the domain of the function?*

   This function is evaluatable as long as the denominator of the fraction is nonzero; thus, the function's domain is the set of values where $x - 3 \neq 0$; in other words, when $x \neq 3$. In interval form, this would be $(-\infty, 3) \cup (3, \infty)$.

   (b) **(2 points)** *Describe, either in words or symbolically, the long-term behavior of the function in each direction.*

   For very large positive, or large-magnitude negative, values of x, the comparative magnitudes of $4x^2$ and 16 will be such as to make 16 a negligible contribution to the numerator; likewise, in the denominator, $x - 3$ will resemble x if x is of extraordinary magnitude. Thus, for very large-magnitude values of x, we see that $f(x) \approx \frac{4x^2}{x} = 4x$, which is a linear function of positive slope. Thus, our reasonable understanding is that for very large positive x, $f(x)$ will likewise be very large and positive; for very large-magnitude negative x, $f(x)$ will also be a negative number of large magnitude. Alternatively, one might write symbolically that

$$\lim_{x\to+\infty} f(x) = +\infty, \quad \lim_{x\to-\infty} f(x) = -\infty$$

6. **(6 points)** *Common stock in the RAMJAC corporation is expected to increase in value by 3% each year. On the basis of this information, we have made a $5000 investment in RAMJAC stock.*

   (a) **(3 points)** *Create a function $f(t)$ to describe the expected value of our investment t years from now.*

   Since the value increases by 3% per year, the value after a year is 103% of the former value; thus, in t years, the investment has a value equal to 1.03^t times the original mass. Thus, $f(t) = 5000\,(1.03^t)$.

   (b) **(3 points)** *We will cash out our investment when it reaches $6500; based on our projection of the stock's change in value, when can we expect this to happen?*

We solve for t when $f(t) = 6500$:

$$\begin{aligned} 5000\left(1.03^t\right) &= 6500 \\ 1.03^t &= \frac{5000}{6500} = 1.3 \\ t &= \log_{1.03} 1.3 = \frac{\ln 1.3}{\ln 1.03} \end{aligned}$$

With a calculator we can ascertain this period to be approximately 8.9 years.

7. **(6 points)** *Let* $f(x) = 2 + x - 3x^2$.

   (a) **(3 points)** *Using the difference quotient, determine the formula for* $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[2 + (x+h) - 3(x+h)^2] - (2 + x - 3x^2)}{h} \\ &= \lim_{h\to 0} \frac{(2 + x + h - 3x^2 - 6xh - 3h^2) - (2 + x - 3x^2)}{h} \\ &= \lim_{h\to 0} \frac{h - 6xh - 3h^2}{h} \\ &= \lim_{h\to 0} 1 - 6x - 3h \\ &= 1 - 6x \end{aligned}$$

   (b) **(3 points)** *Find the equation of the tangent line to* $f(x)$ *at the point* $(1, 0)$.

   We know this line must pass through $(1, 0)$, with slope $g'(1) = 1 - 6 \cdot 1 = -5$. Using point-slope form, we get

$$(y - 0) = -5(x - 1)$$

   which can also be expressed in slope-intercept form as $y = -5x + 5$.

8. **(5 points)** *Let* $f(x) = 5 - 3x$.

   (a) **(1 points)** *Find* $\lim_{x\to 2} f(x)$.

   Since $f(x)$ is a polynomial and thus continuous everywhere, direct substitution makes calculating this limit trivial: $\lim_{x\to 2} f(x) = f(2) = 5 - 3 \cdot 2 = -1$.

   (b) **(4 points)** *Using epsilon-delta methods, justify your result above.*

   Given a value of ϵ, we constrain $f(x)$ to be within ϵ of -1, and attempt to derive a sufficient bound on δ therefrom:

$$\begin{aligned} |5 - 3x - (-1)| &< \epsilon \\ |6 - 3x| &< \epsilon \, |x - 2| < \frac{\epsilon}{3} \end{aligned}$$

   So, since it is sufficient to require x within $\frac{\epsilon}{3}$ of -1, we may establish δ to be $\frac{\epsilon}{3}$.

9. **(3 point bonus)** *Using an appropriate analogue of an epsilon-delta proof, formally prove on the back of this sheet that* $\lim_{x\to-\infty} e^{(x^2)} = +\infty$.

First, let us recall the definition of the statement $\lim_{x\to-\infty} f(x) = +\infty$:

For any number E, there is a number D such that, if $x < D$, then $f(x) > E$

Alternatively, such a statement is a promise that, for any (presumably very large) challenge value E which we are instructed to make $f(x)$ exceed, we can provide an upper bound D on x so that $f(x)$ will indeed meet the challange.

In this specific case, then, we want to find out which values of x are low enough to make e^{x^2} exceed a given value E. We now solve for the value of x yielded by the challenge inequality:

$$\begin{aligned} e^{x^2} &> E \\ \ln e^{x^2} &> \ln E \\ x^2 &> \ln E \\ x &> \sqrt{\ln E} \text{ or } x < -\sqrt{\ln E} \end{aligned}$$

Thus, in order to meet the challenge, either x must be larger than $\sqrt{\ln E}$, or smaller than $-\sqrt{\ln E}$. In this particular case, our control value D grants us the ability only to make x very small; thus, the requirement which we shall meet with our delta-analogue is $x < -\sqrt{\ln E}$; we may do this by letting $D = \sqrt{\ln E}$ (or any smaller value). We have seen then that, if $x < D$, it will indeed be the case that $e^{x^2} > E$, satisfying the requirements for the statement $\lim_{x\to-\infty} e^{(x^2)} = +\infty$ to be true.