1. **(16 points)** *Answer the following derivative-related questions.*

   (a) **(8 points)** *If $y = \tan\sqrt{\sec x}$, find $\frac{dy}{dx}$.*
   This is an application of the chain rule in two stages, in which we shall let $u = \sqrt{\sec x} = \sqrt{v}$, and $v = \sec x$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\tan\sqrt{\sec x} \\
&= \frac{d}{dx}\tan u \\
&= \frac{du}{dx}\frac{d}{du}\tan u \\
&= \left(\frac{d}{dx}\sqrt{v}\right)\frac{d}{du}\tan u \\
&= \frac{dv}{dx}\left(\frac{d}{dv}\sqrt{v}\right)\frac{d}{du}\tan u \\
&= (\sec x\tan x)\cdot\frac{1}{2\sqrt{v}}\cdot\sec^2 u \\
&= \sec x\tan x\cdot\frac{1}{2\sqrt{\sec x}}\cdot\sec^2\sqrt{\sec x}
\end{aligned}$$

   (b) **(4 points)** *Find $\frac{d}{dx}\left((e^x + 2x)\arcsin x\right)$.*
   This is an application of the product rule:

$$\begin{aligned}
\frac{d}{dx}\left((e^x + 2x)\arcsin x\right) &= \frac{d}{dx}(e^x + 2x)\arcsin x + (e^x + 2x)\frac{d}{dx}\arcsin x \\
&= (e^x + 2)\arcsin x +)e^x + 2x)\cdot\frac{1}{\sqrt{1 - x^2}}
\end{aligned}$$

   (c) **(4 points)** *If $g(t) = \frac{\ln t}{\sqrt[3]{t}}$, find $g'(t)$.*
   This is an application of the quotient rule:

$$\begin{aligned}
g'(t) &= \frac{d}{dt}\frac{\ln t}{\sqrt[3]{t}} \\
&= \frac{\sqrt[3]{t}\left(\frac{d}{dt}\ln t\right) - \ln t\frac{d}{dt}t^{1/3}}{\left(\sqrt[3]{t}\right)^2} \\
&= \frac{\sqrt[3]{t}\cdot\frac{1}{t} - \ln t\cdot\frac{1}{3}t^{-2/3}}{\left(\sqrt[3]{t}\right)^2}
\end{aligned}$$

   This problem could alternatively be solved as a product-rule invocation on $g'(t) = \ln t\cdot t^{-1/3}$.

2. **(12 points)** *Calculate $\frac{d}{dx}\cos(x^2e^x)$.*

Let $u = x^2e^x$; then this is a straightforward chain rule invocation, although with a product rule embedded within it:

$$\begin{aligned}
\frac{d}{dx}\cos(x^2e^x) &= \frac{d}{dx}\cos(u) \\
&= \frac{du}{dx}\frac{d}{du}\cos(u) \\
&= \frac{d}{dx}\left(x^2e^x\right)(-\sin u) \\
&= \left(2xe^x + x^2e^x\right)(-\sin u) \\
&= \left(2xe^x + x^2e^x\right)\left(-\sin(x^2e^x)\right)
\end{aligned}$$

3. **(10 points)** *Find an equation of the tangent line to the curve* $y = e^x(x^2 - 3x + 1)$ *at* $(0, 1)$.

Using the product rule,

$$\frac{dy}{dx} = e^x(x^2 - 3x + 1) + e^x(2x - 3)$$

and specifically when $x = 0$, $\frac{dy}{dx} = e^0 \cdot 1 + e^0(-3) = -2$, so the desired line has slope -2, and thus equation $y = -2x + b$. Since it passes through the point $(0, 1)$ we may specifically determine that $1 = -2 \cdot 0 + b$, and thus that $b = 1$, for an equation $y = -2x + 1$ of the tangent line.

4. **(15 points)** *The cissoid of Diocles is a curve satisfying the equation* $x(x^2 + y^2) = 4y^2$.

(a) **(12 points)** *Find a formula for* $\frac{dy}{dx}$ *on this curve.*

Taking the derivative of each side, we get

$$\begin{aligned}
\frac{d}{dx}\left(x(x^2 + y^2)\right) &= \frac{d}{dx}\left(4y^2\right) \\
(x^2 + y^2) + x\frac{d}{dx}(x^2 + y^2) &= \frac{dy}{dx}\frac{d}{dy}\left(4y^2\right) \\
(x^2 + y^2) + 2x^2 + x\frac{d}{dx}y^2 &= \frac{dy}{dx}8y \\
3x^2 + y^2 + x\frac{dy}{dx}\frac{d}{dy}y^2 &= \frac{dy}{dx}8y \\
3x^2 + y^2 + 2xy\frac{dy}{dx} &= 8y\frac{dy}{dx} \\
2xy\frac{dy}{dx} - 8y\frac{dy}{dx} &= -3x^2 - y^2 \\
(2xy - 8y)\frac{dy}{dx} &= -3x^2 - y^2 \\
\frac{dy}{dx} &= \frac{-3x^2 - y^2}{2xy - 8y} = \frac{3x^2 + y^2}{8y - 2xy}
\end{aligned}$$

(b) **(3 points)** *Find the equation of the tangent line to the curve at* $(2, -2)$.

At the specific values $x = 2$, $y = -2$, we see from the above that

$$\frac{dy}{dx} = \frac{3 \cdot 2^2 + (-2)^2}{8(-2) - 2 \cdot 2(-2)} = \frac{16}{-8} = -2$$

Thus we have a line of the form $y = -2x + b$; plugging in $(2, -2)$ we can solve for b:

$$\begin{aligned} -2 &= -2 \cdot 2 + b \\ 2 &= b \end{aligned}$$

so $y = -2x + 2$ is the tangent line.

5. **(9 points)** *Find the absolute maxima and minima of the function* $f(x) = x^3 - 2x^2 - 4x + 3$ *on the interval* $[0, 3]$.

We know that

$$f'(x) = 3x^2 - 4x - 4$$

We then seek out critical points. $f'(x)$ is a polynomial, so points where $f'(x)$ does not exist are not under consideration. We also look for values where $f'(x) = 0$. This occurs when $3x^2 - 4x - 4 = 0$, which, either by factorization or by the quadratic formula can be found to be the case when $x = 2$ or $x = -\frac{2}{3}$. $x = -\frac{2}{3}$ is outside our interval, so the only critical point within the interval is $x = 2$.

Thus, the only candidates for minimum and maximum are the critical point $x = 2$ and the endpoints $x = 0$ and $x = 3$. $f(0) = 0^3 - 2 \cdot 0^2 - 4 \cdot 0 + 3 = 3$, $f(2) = 2^3 - 2 \cdot 2^2 - 4 \cdot 2 + 3 = -5$, and $f(3) = 3^3 - 2 \cdot 3^2 - 4 \cdot 3 + 3 = 0$. Of these, we may thus say with certainty that the highest value (the absolute maximum) is at $x = 0$, and the lowest value (the absolute minimum) at $x = 2$.

6. **(16 points)** *Amy is standing motionless* 50 *meters east of a north-south road with a radar gun, while Bob, who is* 120 *meters to the north, is driving south. The radar gun reports how quickly the* distance between Amy and Bob *is changing (which may not be Bob's actual speed).*

Our setup for this problem involves a right triangle between Bob's car, the place on the road parallel to Amy, and Amy's position well off the road. The east-west leg between Amy and the road is a constant length of 50, while the length of the north-south leg and the length of the hypotenuse are changing. We may denote these respectively by y and s, and note that, by the Pythagorean Theorem, they have the relationship $y^2 + 50^2 = s^2$. We also know that y is *currently* 120, and can compute in additon that s is currently

$$\sqrt{50^2 + y^2} = \sqrt{2500 + 14400} = \sqrt{16900} = 130$$

.

(a) **(12 points)** *If Bob is driving south at* 30 *meters per second, what will the radar report as the rate of change of the distance between Bob and Amy?*

The information given to us here is that $\frac{dy}{dt} = -30$ (since y is decreasing at a rate of 30 meters per second), and that we wish to find $\frac{ds}{dt}$. Using the relationship described in the genral problem, we differentiate both sides with respect to t:

$$\begin{aligned} y^2 + 50^2 &= s^2 \\ \frac{d}{dt}\left(y^2 + 50^2\right) &= \frac{d}{dt}s^2 \\ \frac{dy}{dt}\frac{d}{dy}\left(y^2 + 50^2\right) &= \frac{ds}{dt}\frac{d}{ds}s^2 \\ \frac{dy}{dt}2y &= \frac{ds}{dt}2s \\ \frac{\frac{dy}{dt}y}{s} &= \frac{ds}{dt} \end{aligned}$$

and thus $\frac{ds}{dt} = \frac{-30\cdot120}{130} = \frac{-36}{13}$.

(b) **(4 points)** *Conversely, if the radar reported a change-rate of* 25 *meters per second, what would Bob's actual speed be?*

In this case we are given $\frac{ds}{dt} = \pm 25$, and asked to find $\frac{dy}{dt}$. We may start with the second-to last line of the derviation seen in the previous section:

$$\frac{dy}{dt}2y = \frac{ds}{dt}2s$$

and solve for $\frac{dy}{dt} = \frac{\frac{ds}{dt}s}{y} = \frac{\pm25\cdot130}{120} = \pm\frac{325}{12}$.

7. **(9 points)** *Estimate the following values using appropriate linear approximations.*

(a) **(4 points)** $(-1.993)^4$.

We consider the function $f(x) = x^4$, whose derivative is $f'(x) = 4x^3$. For x close to -2 (as -1.993 is), we can use the linear approximation:

$$f(x) \approx f(-2) + (x+2)f'(-2)$$

Since $f(-2) = 16$ and $f'(-2) = -32$, it follows that

$$f(-1.993) \approx 16 + 0.007(-32) = 16 - 0.224 = 15.776$$

For purposes of comparison, the actual value of $(-1.993)^4$ is around 15.777173.

(b) **(5 points)** $\sqrt{25.07}$.

We consider the function $f(x) = \sqrt{x}$, whose derivative is $\frac{1}{2\sqrt{x}}$. For x close to 25 (as 25.07 is), we can use the linear approximation:

$$f(x) \approx f(25) + (x-25)f'(25)$$

Since $f(25) = 5$ and $f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$, it follows that

$$f(25.07) \approx 5 + (0.07)\cdot\frac{1}{10} = 5.007$$

For purposes of comparison, the actual value of $\sqrt{25.07}$ is around 5.0069951.

8. **(13 points)** *If* $f(x) = \frac{\arctan 3x}{\sqrt{x^4+2}}$*, then find* $f'(x)$*.*

We shall need the quotient rule immediately, and will eventually need recourse to the chain rule; preemptively we shall define $u = 3x$ and $v = x^4 + 2$, which we will need later. Then:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\frac{\arctan u}{\sqrt{v}} \\
&= \frac{\sqrt{v}\left(\frac{d}{dx}\arctan u\right) - \arctan u \frac{d}{dx}\sqrt{v}}{(\sqrt{v})^2} \\
&= \frac{\sqrt{v}\left(\frac{du}{dx}\frac{d}{du}\arctan u\right) - \arctan u \frac{dv}{dx}\frac{d}{dv}\sqrt{v}}{v} \\
&= \frac{\sqrt{v}\left(\frac{d}{dx}3x\right)\cdot\frac{1}{1+u^2} - \arctan u\left(\frac{d}{dx}x^4+2\right)\frac{1}{2\sqrt{v}}}{v} \\
&= \frac{\sqrt{x^4+2}\cdot 3\cdot\frac{1}{1+(3x)^2} - \arctan(3x)\cdot 4x^3\cdot\frac{1}{2\sqrt{x^4+2}}}{x^4+2} \\
&= \frac{\frac{3\sqrt{x^4+2}}{1+9x^2} - \frac{4x^3\arctan(3x)}{2\sqrt{x^4+2}}}{x^4+2}
\end{aligned}$$

The last line of the above calculation is cleanup and is optional.

9. **(6 point bonus)** *Currently Yvette is 10 miles north of the Library of Babel, walking south at 3mph, while Zachary is 1 mile east of the Library, walking east at 5mph. How soon will it be the case that the distance between them is (if only momentarily) unchanging?*

This can be solved either purely as a derivative calculation, or as a related rates problem. The key observation is that we are interested in various properties at times *other* than the present, so it would be erroneous to work under the presumption that Yvette and Zachary's distances from the library are specifically 10 and 1 mile respectively; instead, in t hours, their distances will be $10 - 3t$ and $1 + 5t$ respectively.

To solve this purely as a derivative, one could then compute the distance between them at time t to be

$$s = \sqrt{(10-3t)^2 + (1+5t)^2} = \sqrt{34t^2 - 50t + 101}$$

and then using the chain rule,

$$\frac{ds}{dt} = \frac{68t - 50}{2\sqrt{34t^2 - 50t + 101}}$$

and since we want to know *when* $\frac{ds}{dt}$ is zero, we set this expression to zero and solve for t, getting $t = \frac{50}{68} = \frac{25}{34}$.

Alternatively, we could let Yvette's and Zachary's distances from the library be given by the variables $y = 10 - 3t$ and $z = 1 + 5t$, and by the further known derivatives $\frac{dy}{dt} = -3$ and

$\frac{dz}{dt} = 5$. We know $y^2 + z^2 = s^2$, so differentiating both sides with respect to t:

$$\begin{aligned} \frac{d}{dt}\left(y^2 + z^2\right) &= \frac{d}{dt}s^2 \\ \frac{dy}{dt}\frac{d}{dy}y^2 + \frac{dz}{dt}\frac{d}{dz}z^2 &= \frac{ds}{dt}\frac{d}{ds}s^2 \\ \frac{dy}{dt}2y + \frac{dz}{dt}2z &= \frac{ds}{dt}2s \\ 2 \cdot (-3)(10 - 3t) + 2 \cdot 5(1 + 5t) &= \frac{ds}{dt}2s \end{aligned}$$

and since we are interested in when $\frac{ds}{dt} = 0$, this equation can be simplified to $-50 + 68t = 0$, with solution $t = \frac{25}{34}$.