1. **(8 points)** *A sample of phlebotinum-75 decays over time, losing 23% of its mass per year. We have just obtained a 20-gram sample of this element.*

   (a) **(3 points)** *Create a function $f(t)$ to describe the expected mass of phlebotinum-75 remaining in the sample t years from now.*

   Since the sample decays by 23% per year, the mass after a year is 77% of the former mass; thus, in t years, the sample has a mass equal to 0.77^t of the original mass. Thus, $f(t) = 20\,(0.77^t)$.

   (b) **(5 points)** *The sample is too small to be of further use to us after only 5 grams remain. How long will it take for this to happen?*

   We solve for t when $f(t) = 20$:

$$\begin{aligned} 20\left(0.77^t\right) &= 5 \\ 0.77^t &= \frac{5}{20} = \frac{1}{4} \\ t = \log_{0.77}\frac{1}{4} &= \frac{\ln\frac{1}{4}}{\ln 0.77} \approx 5.3 \text{ years} \end{aligned}$$

2. **(6 points)** *Given the function $f(x) = \frac{(x^2+1)(x+1)}{(x-2)(x+2)(2x+3))}$, answer the following questions preparatory to sketching the functions.*

   (a) **(2 points)** *What is the domain of the function?*

   The denominator is zero when $x - 2 = 0$, $x + 2 = 0$, or $2x + 3 = 0$. These occur at $x = 2$, $x = -2$, and $x = \frac{-3}{2}$ respectively. This prevents the function from being evaluatable at these points. The domain may be given as restriction on x in the form $x \neq 2, -2, \frac{-3}{2}$, or as the interval notation $(-\infty, -2) \cup (-2, \frac{-3}{2}) \cup (\frac{-3}{2}, 2) \cup (2, \infty)$.

   (b) **(2 points)** *Describe, either in words or symbolically, the long-term behavior of the function in each direction.*

   For very large or very negative values of x, $g(x)$ is approximately equal to the quotient of the highest-degree terms in the numerator and denominator. Multiplying out the highest-degree terms in each factor yields $\frac{x^3}{3x^2} = \frac{1}{2}$, so over the long term $g(x)$ tends towards $\frac{1}{2}$. Thus, as $x \to \pm\infty$, $g(x) \to \frac{1}{2}$.

   (c) **(2 points)** *At which x-values does the function equal zero?*

   The function is zero when the numerator is zero and the denominator is not zero. Of the two factors in the numerator, $x^2 + 1$ is never zero, since $x^2 + 1 \geq 1$, but $x + 1$ is zero when $x = -1$. Noting that the denominator is nonzero at $x = -1$, we can conclude that $x = -1$ is a zero of this function.

3. **(6 points)** *Let* $f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 1 \\ \sqrt{x+a} & \text{if } 1 < x \leq 6 \\ bx & \text{if } x > 6 \end{cases}$.

   What choices of a and b will make this function continuous?

   Each of the individual parts of this function can be easily observed to be continuous on its domain, so problems can only arise at the junction points $x = 1$ and $x = 6$. To guarantee continuity at these points, we need to make sure that the left and right limits coincide, as such at $x = 1$:

$$\begin{aligned}\lim_{x\to 1-} f(x) &= \lim_{x\to 1+} f(x)\\ \lim_{x\to 1-} x^2+1 &= \lim_{x\to 1+} \sqrt{x+a}\\ 1^2+1 &= \sqrt{1+a}\\ 2^2 &= 1+a\\ a &= 2^2-1 = 3\end{aligned}$$

And likewise for $x = 6$:

$$\begin{aligned}\lim_{x\to 6-} f(x) &= \lim_{x\to 6+} f(x)\\ \lim_{x\to 6-} \sqrt{x+a} &= \lim_{x\to 6+} bx\\ \sqrt{6+3} &= 6b\\ 3 &= 6b\\ \frac{1}{2} &= b\end{aligned}$$

So our solution is to choose $a = \frac{8}{3}$ and $b = \frac{1}{2}$.

4. **(6 points)** *Let* $g(u) = \frac{2u^2-3u-5}{u+1}$.

   (a) **(2 point)** *Find* $\lim\limits_{u\to -1} g(u)$.

   Note that $g(u) = \frac{(2u-5)(u+1)}{u+1}$. Thus, except at the point $u = -1$, $f(t) = 2u - 5$. Since the limit concerns the behavior not *at* $u = -1$ but in its vicinity, $\lim_{u\to -1} \frac{2u^2-3u-5}{u+1} = \lim_{u\to -1} 2u - 5 = -7$.

   (b) **(4 points)** *Using epsilon-delta methods, justify your result above.*

   Given a value of ϵ, we constrain $g(u)$ to be within ϵ of -7, and attempt to derive a sufficient bound on δ therefrom:

$$\begin{aligned}\left|\frac{2u^2-3u-5}{u+1}+7\right| &< \epsilon\\ \left|\frac{(2u-5)(u+1)}{u+1}+7\right| &< \epsilon\\ |2u-5+7| &< \epsilon \text{ and } u \neq -1\\ |2u+2| &< \epsilon \text{ and } u \neq -1\\ |u+1| &< \frac{\epsilon}{2} \text{ and } u \neq -1\end{aligned}$$

   So, since it is sufficient to require x within $\frac{\epsilon}{2}$ of -1, we may establish δ to be $\frac{\epsilon}{2}$.

5. **(6 points)** *Determine the domains of the following functions:*

(a) **(2 points)** $f(t) = \sqrt{25 - t^2}$.

This function is evaluatable as long as the argument of the square root is non-negative; thus, the function's domain is the set of values there $25 - t^2 \geq 0$; in other words, when $t^2 \leq 25$, or $-5 \leq t \leq 5$. In interval form, this world be $[-5, 5]$.

(b) **(2 points)** $g(s) = \frac{\sqrt{x+1}}{x-3}$.

This function is evaluatable as long as the argument to the square root is non-negative, and the denominator of the fraction is nonzero. Thus, we require that $x + 1 \geq 0$ and $x - 3 \neq 0$; in other words, $x \geq -1$ and $x \neq 3$. In interval form, this could alternatively be written as $[-1, 3) \cup (3, \infty)$.

(c) **(2 points)** $h(x) = \ln(6 - 2x) + \frac{1}{3x-18}$.

This function is evaluatable as long as the argument to the logarithm is positive and the denominator of the fraction is nonzero. Thus, we require that $6-2x > 0$ and $3x-18 \neq 0$, which we simplify to $x < 3$ and $x \neq 6$. The latter condition ends up being moot, since if $x < 3$, $x \neq 6$ necessarily follows. We can thus simplify our condition to $x < 3$ alone, or, in interval notation, $(-\infty, 3)$.

6. **(7 points)** *Let* $g(x) = -3x^2 + 7x - 2$.

(a) **(4 points)** *Using the difference quotient, find* $g'(x)$.

$$\begin{aligned}
g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{[-3(x+h)^2 + 7(x+h) - 2] - (-3x^2 + 7x - 2)}{h} \\
&= \lim_{h \to 0} \frac{-3x^2 - 6xh - 3h^2 + 7x + 7h - 2 - (-3x^2 + 7x - 2)}{h} \\
&= \lim_{h \to 0} \frac{-6xh - 3h^2 + 7h}{h} \\
&= \lim_{h \to 0} -6x - 3h + 7 \text{ justified since } h \neq 0 \\
&= -6x + 7
\end{aligned}$$

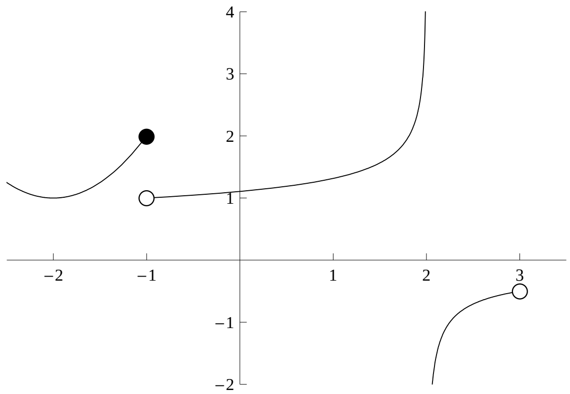
(b) **(3 points)** *Find the equation of the tangent line to* $g(x)$ *at the point* $(2, 0)$.

We know this line must pass through $(2, 0)$, with slope $g'(2) = -6 \cdot 2 + 7 = -5$. Using point-slope form, we get

$$(y - 0) = -5(x - 2)$$

which can also be expressed in slope-intercept form as $y = -5x + 10$.

7. **(7 points)** *For the plot of* $h(x)$ *shown below, indicate whether or not each of the following quantities can be evaluated. If they can be evaluated, compute their values. If they cannot be evaluated, explicitly say so.*



$\lim_{x \to -2^-} h(x) = 1$, since to the left of -2, the graph is very close to the height $y = 1$.

$h(-2) = 1$, as can be observed on the graph.

$\lim_{x \to -1} h(x)$ does not exist, since the left-side and right-side limits at -1 are not equal; they are 2 and 1 respectively.

$h(-1) = 2$; note the solid dot at $(-1, 2)$.

$\lim_{x \to 2^+} h(x)$ does not exist, and specifically the function decreases without bound as$x$ approaches 2 from above; note that slightly to the right of $x = 2$, the function is very negative.

$\lim_{x \to 3^-} h(x) = \frac{1}{2}$, since to the left of 3, the graph is very close to the height $y = \frac{1}{2}$.

$h(3)$ does not exist; at $x = 3$ there is an open dot, which represents an exclusion from the function, at $(3, \frac{1}{2})$, but there are no closed dots to represent an actual value taken on by the function.

8. **(8 points)** *Evaluate the following limits; when a limit can not be evaluated, explain why or describe its behavior.*

   (a) **(2 points)** $\lim_{t \to +\infty} \frac{2t^3-4t^2+7}{-t^4+5t^2-2}$.

   In the long term this function is dominated by its highest-degree terms in the numerator and denominator, so $\lim_{t\to+\infty} \frac{2t^3-4t^2+7}{-t^4+5t^2-2} = \lim_{t\to+\infty} \frac{2t^3}{-t^4} = \lim_{t\to+\infty} \frac{-2}{t} = 0$.

   (b) **(2 points)** $\lim_{\theta \to \pi^-} \sin\theta$.

   Since the sine function is continuous throughout, $\lim_{\theta\to\pi^-} \sin\theta = \sin\pi = 0$.

   (c) **(2 points)** $\lim_{x \to 3} \frac{x^2-x-6}{x-3}$.

   Since we look near, but not at, $x = 1$, we can justify the cancellation $\lim_{x\to3} \frac{x^2-x-6}{x-3} = \lim_{x\to3} \frac{(x-3)(x+2)}{x-3} = \lim_{x\to3} x + 2 = 5$.

   (d) **(2 points)** $\lim_{r \to 3} \frac{r^3-1}{r-3}$.

   This rational function has a zero denominator but not a zero numerator at $r = 3$, so it has an infinite discontinuity there and thus the limit is unevaluatable.