1. **(6 pts)** Below is the graph of a function $f(x)$. For each of the six quantities listed to the right, either give its value or specifically state that it does not exist.

   \[ \lim_{{x \to -3}} f(x) = 4, \text{ since } f(x) \text{ as shown is continuous at } x = -3, \text{ and is seen to be passing through the point } (-3, 4). \]

   \[ \lim_{{x \to -2^+}} f(x) = 4, \text{ since to the right of } -2, \text{ the graph is very close to the height } y = 4. \]

   \[ \lim_{{x \to -2^-}} f(x) = 1, \text{ since to the left of } -2, \text{ the graph is very close to the height } y = 1. \]

   \[ f(-2) = 4; \text{ note the solid dot at } (-2, 4). \]

   \[ \lim_{{x \to 3}} f(x) \text{ does not exist, although specifically the function increases without bound as } x \text{ approaches } 3; \text{ note that slightly to the left and right of } x = 3, \text{ the function takes on very large values.} \]

   \[ \lim_{{x \to 5}} f(x) = 3, \text{ since even though } (5, 3) \text{ is specifically excluded from the function, values of } x \text{ very close to } 5 \text{ yield values of } f(x) \text{ close to } 3. \]

2. **(6 pts)** For each of the following limits, either determine its value or demonstrate that the limit does not exist.

   (a) \[ \lim_{{x \to 1^+}} \frac{2x - 2}{x^3 + 3x - 7}. \]

   \[ \text{Note that direct evaluation of the rational expression } \frac{2 \cdot 1 - 2}{1^3 + 3 \cdot 1 - 7} \text{ yields } \frac{0}{-3} = 0; \text{ since the denominator is nonzero, we may simply invoke this direct evaluation to determine the limit of this rational expression:} \]

   \[ \lim_{{x \to 1^+}} \frac{2x - 2}{x^3 + 3x - 7} = \frac{2 \cdot 1 - 2}{1^3 + 3 \cdot 1 - 7} = \frac{0}{-3} = 0 \]

   (b) \[ \lim_{{t \to 2}} \sqrt{4 - 2t}. \]

   \[ \text{Observe that } \sqrt{4 - 2t} \text{ has domain } (-\infty, 2], \text{ since the square root of a negative number cannot be calculated within the constraints being used in this class; thus, the expression } \sqrt{4 - 2t} \text{ is unevaluable at values of } t \text{ even slightly larger than } 2, \text{ so this limit does not exist.} \]

   (c) \[ \lim_{{s \to -3}} \frac{s^2 + s - 6}{s^2 + 3s}. \]

   \[ \text{Note that direct evaluation of the rational expression } \frac{(-3)^2 + (-3) - 6}{(-3)^2 + 3(-3)} \text{ yields } \frac{0}{0}, \text{ as we have seen, rational expressions of this form are susceptible to factorization and simplification, and in this case we should strive to factor out the term } (s - (-3)), \text{ or more familiarly} \]

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(s + 3) from the numerator and denominator. Use of any factorization (or long/synthetic division) technique will yield

\[
\lim_{s \to -3} \frac{s^2 + s - 6}{s^2 + 3s} = \lim_{s \to -3} \frac{(s - 2)(s + 3)}{s(s + 3)} = \lim_{s \to -3} \frac{s - 2}{s} = \frac{-3 - 2}{-3} = \frac{3}{5}
\]

3. **(4 pts)** Use epsilon-delta methods to prove that \( \lim_{x \to 2} -5x + 2 = -8 \).

We wish to show that there is a value of \( \delta \) such that \(|(-5x + 2) - (-8)| < \epsilon \) follows from \( 0 < |x - 2| < \delta \). We shall start with our desired consequence, and work our way back to the necessary condition via reversible arithmetic steps:

\[
\begin{align*}
|(-5x + 2) - (-8)| &< \epsilon \\
|-5x + 10| &< \epsilon \\
|-5(x - 2)| &< \epsilon \\
|5| &< \epsilon \\
|x - 2| &< \frac{\epsilon}{5}
\end{align*}
\]

Since our desired consequence was equivalent to the condition that \( |x - 2| < \frac{\epsilon}{5} \), and we have the power to make \( |x - 2| \) bounded above by any \( \delta \) we wish, the choice of \( \delta = \frac{\epsilon}{5} \) suffices to guarantee our desired distance between \(-5x + 2\) and \(-8\) (as would any smaller \( \delta \), actually).

4. **(4 pts)** Find \( a \) and \( b \) to make the function \( f(x) = \begin{cases} 
2^x & \text{if } x \leq 3 \\
\frac{a}{x} & \text{if } 3 < x < 8 \\
2x + b & \text{if } x \geq 8
\end{cases} \) continuous everywhere.

Each of the individual parts of this function can be easily observed to be continuous on its domain, so problems can only arise at the junction points \( x = 3 \) and \( x = 8 \). To guarantee continuity at these points, we need to make sure that the left and right limits coincide, as such at \( x = 3 \):

\[
\begin{align*}
\lim_{x \to 3^-} f(x) &= \lim_{x \to 3^+} f(x) \\
\lim_{x \to 3^-} 2^x &= \lim_{x \to 3^+} \frac{a}{x} \\
2^3 &= \frac{a}{\frac{3}{2}} \\
24 &= 3 \cdot 2^3 = a
\end{align*}
\]
and then at \( x = 8 \):

\[
\lim_{x \to 8^-} f(x) = \lim_{x \to 8^+} f(x)
\]

\[
\lim_{x \to 8^-} \frac{a}{x} = \lim_{x \to 8^+} 2x + b
\]

\[
\frac{24}{8} = 2 \cdot 8 + b
\]

\[
3 = 16 + b
\]

\[-13 = b
\]

5. (2 pt bonus) When \( x \) is a nonzero rational number which is written in lowest terms as \( \frac{p}{q} \), let \( f(x) = \frac{1}{p} \); when \( x \) is irrational, let \( f(x) = 0 \). Find, with justification, the value of \( \lim_{x \to 0} f(x) \).

Due to an error setting this down, this one actually has no solution, but the arguments that it doesn’t have one are somewhat nuanced. Consider any \( \delta > 0 \). Since \( \delta \) is positive, so is \( \frac{1}{\delta} \), and let \( N \) be a positive integer such that \( N > \frac{1}{\delta} \), so \( 0 < \frac{1}{N} < \delta \). Since \( \frac{1}{N} \) is rational and in lowest terms, \( f\left(\frac{1}{N}\right) = \frac{1}{p} = 1 \). Now consider a positive integer \( M \) such that \( M > \frac{\pi}{\delta} \). Then \( 0 < \frac{\pi}{M} < \delta \), but since \( \frac{\pi}{M} \) is irrational, \( f\left(\frac{\pi}{M}\right) = 0 \). Thus, regardless of what \( \delta \) is, the interval \((-\delta, \delta)\) contains both values of \( x \) such that \( f(x) = 0 \) and \( f(x) = 1 \). Thus, no \( \delta \) could satisfy a condition \( |f(x) - L| < 0.5 \) around the point \( a = 0 \), since every choice of \( \delta \) will admit values of \( f(x) \) ranging from 0 to 1.