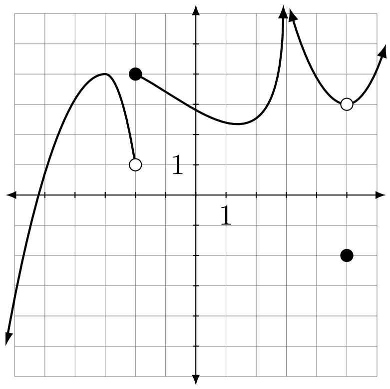
1. **(6 pts)** *Below is the graph of a function $f(x)$. For each of the six quantities listed to the right, either give its value or specifically state that it does not exist.*



$\lim_{x\to -3} f(x) = 4$, since $f(x)$ as shown is continuous at $x = -3$, and is seen to be passing through the point $(-3, 4)$.

$\lim_{x\to -2^+} f(x) = 4$, since to the right of -2, the graph is very close to the height $y = 4$.

$\lim_{x\to -2^-} f(x) = 1$, since to the left of -2, the graph is very close to the height $y = 1$.

$f(-2) = 4$; note the solid dot at $(-2, 4)$.

$\lim_{x\to 3} f(x)$ does not exist, although specifically the function increases without bound as x approaches 3; note that slightly to the left and right of $x = 3$, the function takes on very large values.

$\lim_{x\to 5} f(x) = 3$, since even though $(5, 3)$ is specifically excluded from the function, values of x very close to 5 yield values of $f(x)$ close to 3.

2. **(6 pts)** *For each of the following limits, either determine its value or demonstrate that the limit does not exist.*

(a) $\lim_{x\to 1^+} \frac{2x-2}{x^3+3x-7}$.

Note that direct evaluation of the rational expression $\frac{2\cdot 1-2}{1^3+3\cdot 1-7}$ yields $\frac{0}{-3}$; since the denominator is nonzero, we may simply invoke this direct evaluation to determine the limit of this rational expression:

$$\lim_{x\to 1^+} \frac{2x-2}{x^3+3x-7} = \frac{2\cdot 1-2}{1^3+3\cdot 1-7} = \frac{0}{-3} = 0$$

(b) $\lim_{t\to 2} \sqrt{4-2t}$.

Observe that $\sqrt{4-2t}$ has domain $(-\infty, 2]$, since the square root of a negative number cannot be calculated within the constraints being used in this class; thus, the expression $\sqrt{4-2t}$ is unevaluatable at values of t even slightly larger than 2, so this limit does not exist.

(c) $\lim_{s\to -3} \frac{s^2+s-6}{s^2+3s}$.

Note that direct evaluation of the rational expression $\frac{(-3)^2+(-3)-6}{(-3)^2+3(-3)}$ yields $\frac{0}{0}$; as we have seen, rational expressions of this form are susceptible to factorization and simplification, and in this case we should strive to factor out the term $(s-(-3))$, or more familiarly

$(s+3)$ from the numerator and denominator. Use of any factorization (or long/synthetic division) technique will yield

$$\lim_{s\to -3} \frac{s^2+s-6}{s^2+3s} = \lim_{s\to -3} \frac{(s-2)(s+3)}{s(s+3)} = \lim_{s\to -3} \frac{s-2}{s} = \frac{-3-2}{-3} = \frac{3}{5}$$

3. **(4 pts)** Use epsilon-delta methods to prove that $\lim_{x\to 2} -5x+2 = -8$.

We wish to show that there is a value of δ such that $|(-5x+2)-(-8)| < \epsilon$ follows from $0 < |x-2| < \delta$. We shall start with our desired consequence, and work our way back to the necessary condition via reversible arithmetic steps:

$$\begin{aligned} |(-5x+2)-(-8)| &< \epsilon \\ |-5x+10| &< \epsilon \\ |-5(x-2)| &< \epsilon \\ |-5|\cdot|x-2| &< \epsilon \\ 5|x-2| &< \epsilon \\ |x-2| &< \frac{\epsilon}{5} \end{aligned}$$

Since our desired consequence was equivalent to the condition that $|x-2| < \frac{\epsilon}{5}$, and we have the power to make $|x-2|$ bounded above by any δ we wish, the choice of $\delta = \frac{\epsilon}{5}$ suffices to guarantee our desired distance between $-5x+2$ and 8 (as would any smaller δ, actually).

4. **(4 pts)** Find a and b to make the function $f(x) = \begin{cases} 2^x \text{ if } & x \le 3 \\ \frac{a}{x} \text{ if } & 3 < x < 8 \\ 2x+b \text{ if } & x \ge 8 \end{cases}$ continuous everywhere.

Each of the individual parts of this function can be easily observed to be continuous on its domain, so problems can only arise at the junction points $x=3$ and $x=8$. To guarantee continuity at these points, we need to make sure that the left and right limits coincide, as such at $x=3$:

$$\begin{aligned} \lim_{x\to 3-} f(x) &= \lim_{x\to 3+} f(x) \\ \lim_{x\to 3-} 2^x &= \lim_{x\to 3+} \frac{a}{x} \\ 2^3 &= \frac{a}{3} \\ 24 = 3\cdot 2^3 &= a \end{aligned}$$

and then at $x = 8$:

$$\begin{aligned}
\lim_{x\to 8-} f(x) &= \lim_{x\to 8+} f(x)\\
\lim_{x\to 8-} \frac{a}{x} &= \lim_{x\to 8+} 2x+b\\
\frac{24}{8} &= 2\cdot 8+b\\
3 &= 16+b\\
-13 &= b
\end{aligned}$$

5. **(2 pt bonus)** *When x is a nonzero rational number which is written in lowest terms as $\frac{p}{q}$, let $f(x) = \frac{1}{p}$; when x is irrational, let $f(x) = 0$. Find, with justification, the value of* $\lim_{x\to 0} f(x)$.

   Due to an error setting this down, this one actually has no solution, but the arguments that it doesn't have one are somewhat nuanced. Consider *any* $\delta > 0$. Since δ is positive, so is $\frac{1}{\delta}$, and let N be a positive integer such that $N > \frac{1}{\delta}$, so $0 < \frac{1}{N} < \delta$. Since $\frac{1}{N}$ is rational and in lowest terms, $f(\frac{1}{N}) = \frac{1}{1} = 1$. Now consider a positive integer M such that $M > \frac{\pi}{\delta}$. Then $0 < \frac{\pi}{M} < \delta$, but since $\frac{\pi}{M}$ is irrational, $f(\frac{\pi}{M}) = 0$. Thus, regraadless of what δ is, the interval $(-\delta, \delta)$ contains both values of x such that $f(x) = 0$ and $f(x) = 1$! Thus, no δ could satisfy a condition $|f(x) - L| < 0.5$ around the point $a = 0$, since every choice of δ will admit values of $f(x)$ ranging from 0 to 1.