

1. **(6 points)** Compute the following infinite limits, or, if the limit is uncomputable, briefly describe (in symbols or words) the relevant long-term behavior of the function.

(a) **(2 points)** $\lim_{x \rightarrow -\infty} \frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}$.

Using the highest-order-term technique, we can be certain that for values of x of very large magnitude (large negative magnitude, in this case), $\frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5} \approx \frac{4x^3}{2x^3} = 2$. Thus,

$$\lim_{x \rightarrow -\infty} \frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5} = 2.$$

Alternatively, using the technique of division by the highest degree in the denominator,

$$\lim_{x \rightarrow -\infty} \frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5} = \lim_{x \rightarrow -\infty} \frac{\frac{4x^3 + 6x^2 - 2}{x^3}}{\frac{2x^3 - 4x + 5}{x^3}} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{6}{x} - \frac{2}{x^3}}{2 - \frac{4}{x^2} + \frac{5}{x^3}} = \frac{4 + 0 - 0}{2 - 0 + 0} = 2$$

(b) **(2 points)** $\lim_{t \rightarrow +\infty} \frac{(t-2)(t+5)}{t^2(2t-6)}$.

Using the highest-order-term technique, we can be certain that for values of t of very large magnitude (large positive magnitude, in this case), $\frac{(t-2)(t+5)}{t^2(2t-6)} = \frac{t^2 + 3t - 10}{2t^3 - 6t^2} \approx \frac{t^2}{2t^3} = \frac{1}{2t}$. Thus,

$$\lim_{t \rightarrow +\infty} \frac{(t-2)(t+5)}{t^2(2t-6)} = \lim_{t \rightarrow +\infty} \frac{1}{2t} = 0.$$

Alternatively, using the technique of division by the highest degree in the denominator,

$$\lim_{t \rightarrow +\infty} \frac{(t-2)(t+5)}{t^2(2t-6)} = \lim_{t \rightarrow +\infty} \frac{\frac{(t-2)(t+5)}{t^3}}{\frac{t^2(2t-6)}{t^3}} = \lim_{t \rightarrow +\infty} \frac{\left(1 - \frac{2}{t}\right)\left(\frac{1}{t} + \frac{5}{t^2}\right)}{2 - \frac{6}{t}} = \frac{(1-0)(0+0)}{2-0} = 0$$

(c) **(2 points)** $\lim_{r \rightarrow -\infty} \frac{2-r^3}{r^2+2r+1}$.

Using the highest-order-term technique, we can be certain that for values of r of very large magnitude (large negative magnitude, in this case), $\frac{2-r^3}{r^2+2r+1} \approx \frac{-r^3}{r^2} = -r$. Thus,

$\lim_{r \rightarrow -\infty} \frac{2-r^3}{r^2+2r+1} = \lim_{r \rightarrow -\infty} -r$. Note that this limit does not actually exist but follows an easy to describe behavior: as r becomes a very large-magnitude negative number, $-r$, and by inference $\frac{2-r^3}{r^2+2r+1}$ will necessarily be a positive number of equally enormous magnitude; thus, we could say that $\frac{2-r^3}{r^2+2r+1}$ increases without bound, or if we wished to write it purely symbolically, $\lim_{r \rightarrow -\infty} \frac{2-r^3}{r^2+2r+1} = +\infty$.

Alternatively, using the technique of division by the highest degree in the denominator,

$$\lim_{r \rightarrow -\infty} \frac{2-r^3}{r^2+2r+1} = \lim_{r \rightarrow -\infty} \frac{\frac{2-r^3}{r^2}}{\frac{r^2+2r+1}{r^2}} = \lim_{r \rightarrow +\infty} \frac{\frac{2}{r^2} - r}{1 + \frac{2}{r} + \frac{1}{r^2}}$$

and as $r \rightarrow -\infty$, we see that several of these terms dwindle towards zero, leaving us with $\frac{0-r}{1+0+0}$; we then proceed to analyze its behavior as in the other method.

2. **(4 points)** Using the difference quotient, find the derivative of the function $f(x) = 4x^2 - 5x$.

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[4(x+h)^2 - 5(x+h)] - (4x^2 - 5x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(4x^2 + 8xh + 4h^2 - 5x - 5h) - (4x^2 - 5x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{8xh + 4h^2 - 5h}{h} \\
 &= \lim_{h \rightarrow 0} 8x + 4h - 5 \\
 &= 8x - 5
 \end{aligned}$$

3. (4 points) If $g(t) = 6t^3 - \frac{1}{\sqrt{t}} + 6e^t$, compute $g''(t)$.

Let us start by recasting $g(t)$ in a form which is amenable to differentiation; i.e. rewrite the square root specifically as a power function:

$$g(t) = 6t^3 - t^{-1/2} + 6e^t$$

and now we can take the derivative of this expression to get $g'(t)$:

$$g'(t) = \frac{d}{dt}g(t) = \frac{d}{dt}(6t^3 - t^{-1/2} + 6e^t) = 18t^2 + \frac{1}{2}t^{-3/2} + 6e^t$$

and we get the second derivative by once again differentiating:

$$g''(t) = \frac{d}{dt}g'(t) = \frac{d}{dt}\left(18t^2 + \frac{1}{2}t^{-3/2} + 6e^t\right) = 36t^2 - \frac{3}{4}t^{-5/2} + 6e^t$$

Note that the middle term could alternatively be written, if desired, as $-\frac{3}{4\sqrt{t^5}}$.

4. (6 points) Perform the following computations; you do not need to algebraically simplify fully differentiated expressions.

- (a) (3 points) Determine $\frac{d}{ds} \frac{s^3 - 3s}{5s^2 - e^s}$.

We invoke the quotient rule:

$$\begin{aligned}
 \frac{d}{ds} \frac{s^3 - 3s}{5s^2 - e^s} &= \frac{(5s^2 - e^s) \frac{d}{ds}(s^3 - 3s) - (s^3 - 3s) \frac{d}{ds}(5s^2 - e^s)}{(5s^2 - e^s)^2} \\
 &= \frac{(5s^2 - e^s)(3s^2 - 3) - (s^3 - 3s)(10s^2 - e^s)}{(5s^2 - e^s)^2}
 \end{aligned}$$

And further simplification is, as mentioned, not necessary.

- (b) (3 points) If $y = (t^e + e^t) \left(3\sqrt{t} - \frac{1}{t^3}\right)$, find $\frac{dy}{dt}$.

Noting that $\sqrt{t} = t^{1/2}$ and $\frac{1}{t^3} = t^{-3}$, we invoke the quotient rule:

$$\begin{aligned}
 \frac{dy}{dt} &= \frac{d}{dt} \left[(t^e + e^t) \left(3\sqrt{t} - \frac{1}{t^3}\right) \right] \\
 &= \left[\frac{d}{dt}(t^e + e^t) \right] \left(3\sqrt{t} - \frac{1}{t^3}\right) + (t^e + e^t) \frac{d}{dt} \left(3t^{1/2} - t^{-3}\right) \\
 &= (et^{e-1} + e^t) \left(3\sqrt{t} - \frac{1}{t^3}\right) + (t^e + e^t) \left(\frac{3}{2}t^{-1/2} + 3t^{-4}\right)
 \end{aligned}$$

And further simplification is, as mentioned, not necessary.

5. **(2 point bonus)** If $f(x) = x^2e^x$, find (with justification) a general formula for $f^{(n)}(x)$ on the back of this page.

We might pre-emptively note, by using the product rule, that $\frac{d}{dx}(x^2e^x) = x^2e^x + 2xe^x$ and that $\frac{d}{dx}(xe^x) = xe^x + e^x$; additionally, we see that $\frac{d}{dx}e^x = e^x$ as we already knew. We thus reasonably expect that repeated differentiation of x^2e^x will spawn terms of the form x^2e^x , xe^x , and e^x , since only terms of those forms result from taking the derivatives of terms of those forms.

If we perform a few simple derivatives, we can see patterns emerging.

$$\begin{aligned}
 f(x) &= x^2e^x & &= x^2e^x \\
 f'(x) &= x^2e^x + 2xe^x & &= x^2e^x + 2xe^x \\
 f''(x) &= x^2e^x + 2xe^x + 2xe^x + 2e^x & &= x^2e^x + 4xe^x + 2e^x \\
 f'''(x) &= x^2e^x + 2xe^x + 4xe^x + 4e^x + 2e^x & &= x^2e^x + 6xe^x + (2+4)e^x \\
 f^{(4)}(x) &= x^2e^x + 2xe^x + 6xe^x + 6e^x + (2+4)e^x & &= x^2e^x + 8xe^x + (2+4+6)e^x
 \end{aligned}$$

and we could form a conjecture (which is not too hard to prove, although it would be easiest to do using the principle of induction, which is not covered in this class), that

$$f^{(n)}(x) = x^2e^x + nxe^x + (2 + 4 + \cdots + 2(n-1))e^x$$

and, using rules for simplifying arithmetic series, the last term can actually be folded into a closed form to yield:

$$f^{(n)}(x) = x^2e^x + nxe^x + n(n-1)e^x$$