1. **(12 points)** *Compute the following derivatives:*

   (a) **(4 points)** *Given $y = x^2 e^{\sec x}$, find $\frac{dy}{dx}$.*
   This is a product rule stacked with a chain rule (since $e^{\sec x}$ is a composition). Let us pre-emptively assign $u = \sec x$, so that

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(x^2 e^{\sec x}\right) \\
&= \left(\frac{d}{dx}x^2\right)e^{\sec x} + x^2\frac{d}{dx}e^u \\
&= 2xe^{\sec x} + x^2\frac{du}{dx}\frac{d}{du}e^u \\
&= 2xe^{\sec x} + x^2(\sec x\tan x)e^u \\
&= 2xe^{\sec x} + x^2\sec x\tan x e^{\sec x}
\end{aligned}$$

   (b) **(4 points)** *For $f(t) = \cos\left(\frac{e^t-2}{\arcsin t}\right)$, find $f'(t)$.*
   This is a chain rule problem whose internal calculation is itself a quotient rule. Let us pre-emptively assign $u = \frac{e^t-2}{\arcsin t}$, and then

$$\begin{aligned}
f'(t) &= \frac{d}{dt}\cos\left(\frac{e^t-2}{\arcsin t}\right) \\
&= \frac{d}{dt}\cos u \\
&= \frac{du}{dt}\frac{d}{du}\cos u \\
&= \left(\frac{d}{dt}\frac{e^t-2}{\arcsin t}\right)\frac{d}{du}\cos u \\
&= \left(\frac{\arcsin t\frac{d}{dt}(e^t-2) - (e^t-2)\frac{d}{dt}\arcsin t}{(\arcsin t)^2}\right)(-\sin u) \\
&= \left(\frac{(\arcsin t)e^t - (e^t-2)\frac{1}{\sqrt{1-t^2}}}{(\arcsin t)^2}\right)\left(-\sin\frac{e^t-2}{\arcsin t}\right)
\end{aligned}$$

   (c) **(4 points)** *Compute $\frac{d}{d\theta}\sin(\theta + \ln\theta)$.*
   This is a chain rule problem; let $u = \theta + \ln\theta$, and then:

$$\begin{aligned}
\frac{d}{d\theta}\sin(\theta+\ln\theta) &= \frac{d}{d\theta}\sin u \\
&= \frac{du}{d\theta}\frac{d}{du}\sin u \\
&= \left(1+\frac{1}{\theta}\right)\cos u \\
&= \left(1+\frac{1}{\theta}\right)\cos(\theta+\ln\theta)
\end{aligned}$$

2. **(4 points)** *Given the equality* $x \sin y = 1 - y^2$*, find a formula for* $\frac{dy}{dx}$ *by implicit differentiation.*

   Differentiating both sides of the equation with respect to x:

$$\begin{aligned}
\frac{d}{dx}(x \sin y) &= \frac{d}{dx}\left(1 - y^2\right) \\
\left(\frac{d}{dx}x\right)\sin y + x\frac{d}{dx}\sin y &= 0 - \frac{d}{dx}y^2 \\
\sin y + x\frac{dy}{dx}\frac{d}{dy}\sin y &= -\frac{dy}{dx}\frac{d}{dy}y^2 \\
\sin y + x\frac{dy}{dx}\cos y &= -\frac{dy}{dx}2y \\
x\frac{dy}{dx}\cos y + 2y\frac{dy}{dx} &= -\sin x \\
(x\cos y + 2y)\frac{dy}{dx} &= -\sin x \\
\frac{dy}{dx} &= \frac{-\sin x}{x\cos y + 2y}
\end{aligned}$$

3. **(4 points)** *An air compressor which delivers 10 cubic feet per minute is being used to inflate a spherical balloon of volume* V *and radius* r *which is currently 8 feet in radius; we wish to know how quickly the radius is increasing. Calculate* $\frac{dr}{dt}$ *below, using the facts:* $V = \frac{4}{3}\pi r^3$*,* $\frac{dV}{dt} = 10$*,* $r = 8$ *currently.*

   We differentiate both sides of the expression $V = \frac{4}{3}\pi r^3$ with respect to t, and then algebraically isolate $\frac{dr}{dt}$.

$$\begin{aligned}
\frac{d}{dt}V &= \frac{d}{dt}\frac{4}{3}\pi r^3 \\
\frac{dV}{dt} &= \frac{dr}{dt}\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \\
\frac{dV}{dt} &= \frac{dr}{dt}4\pi r^2 \\
\frac{\frac{dV}{dt}}{4\pi r^2} &= \frac{dr}{dt}
\end{aligned}$$

   and since $\frac{dV}{dt} = 10$ and the current value of r is 8, we may conclude that the current value of $\frac{dr}{dt}$ is $\frac{10}{4\pi(8^2)} = \frac{10}{256\pi} = \frac{5}{128\pi}$.

4. **(2 point bonus)** *If* n *is a positive integer, find a general formula for* $\frac{d^n}{dx^n}\left(x^{n-1}\ln x\right)$ *on the back of this page.*

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We might look at some specific examples, deriving each example from its predecessor:

$$\begin{aligned}
\frac{d}{dx}\ln x &= \frac{1}{x} \\
\frac{d^2}{dx^2}(x\ln x) &= \frac{d}{dx}\left(\ln x + \frac{x}{x}\right) = \frac{d}{dx}(\ln x + 1) \\
&= \frac{1}{x} \\
\frac{d^3}{dx^3}(x^2\ln x) &= \frac{d^2}{dx^2}\left(2x\ln x + \frac{x^2}{x}\right) = \frac{d^2}{dx^2}(2x\ln x + x) \\
&= 2\cdot\frac{1}{x} = \frac{2}{x} \\
\frac{d^4}{dx^4}(x^3\ln x) &= \frac{d^3}{dx^3}\left(3x^2\ln x + \frac{x^3}{x}\right) = \frac{d^3}{dx^3}(3x^2\ln x + x^2) \\
&= 3\cdot\frac{2}{x} = \frac{6}{x} \\
\frac{d^5}{dx^5}(x^4\ln x) &= \frac{d^4}{dx^4}\left(4x^3\ln x + \frac{x^4}{x}\right) = \frac{d^4}{dx^4}(3x^3\ln x + x^3) \\
&= 4\cdot\frac{6}{x} = \frac{24}{x}
\end{aligned}$$

and the pattern that emerges is that after taking one of the n derivatives requested of $x^{n-1}\ln x$, we are left with $\frac{d^{n-1}}{dx^{n-1}}\left((n-1)x^{n-2}\ln x + x^{n-2}\right)$. The second term of this sum can be easily seen to be zero — hit x^{n-2} with $n-1$ derivatives and it will reach zero — while the first term of this sum looks a lot like the original value we were looking at, but with all exponents decreased by 1. If we repeat this process $n-1$ times, we would get that

$$\frac{d^n}{dx^n}\left(x^{n-1}\ln x\right) = \frac{(n-1)(n-2)(n-3)\cdots 2\cdot 1}{x}$$