1. **(12 points)** Compute the following derivatives:

(a) **(4 points)** Given \( y = x^2 e^{\sec x} \), find \( \frac{dy}{dx} \).

This is a product rule stacked with a chain rule (since \( e^{\sec x} \) is a composition). Let us pre-emptively assign \( u = \sec x \), so that

\[
\frac{dy}{dx} = \frac{d}{dx} \left( x^2 e^{\sec x} \right) \\
= \left( \frac{d}{dx} x^2 \right) e^{\sec x} + x^2 \frac{d}{dx} e^u \\
= 2xe^{\sec x} + x^2 \frac{du}{dx} \frac{d}{du} e^u \\
= 2xe^{\sec x} + x^2 (\sec x \tan x)e^u \\
= 2xe^{\sec x} + x^2 \sec x \tan x e^{\sec x}
\]

(b) **(4 points)** For \( f(t) = \cos \left( \frac{e^t - 2}{\arcsin t} \right) \), find \( f'(t) \).

This is a chain rule problem whose internal calculation is itself a quotient rule. Let us pre-emptively assign \( u = \frac{e^t - 2}{\arcsin t} \), and then

\[
f'(t) = \frac{d}{dt} \cos \left( \frac{e^t - 2}{\arcsin t} \right) \\
= \frac{d}{dt} \cos u \\
= \frac{du}{dt} \frac{d}{du} \cos u \\
= \left( \frac{\arcsin t \frac{d}{dt} (e^t - 2) - (e^t - 2) \frac{d}{dt} \arcsin t}{(\arcsin t)^2} \right) (-\sin u) \\
= \left( \frac{\arcsin t e^t - (e^t - 2) \frac{1}{\sqrt{1-t^2}}}{(\arcsin t)^2} \right) (-\sin \left( \frac{e^t - 2}{\arcsin t} \right))
\]

(c) **(4 points)** Compute \( \frac{d}{d\theta} \sin(\theta + \ln \theta) \).

This is a chain rule problem; let \( u = \theta + \ln \theta \), and then:

\[
\frac{d}{d\theta} \sin(\theta + \ln \theta) = \frac{d}{d\theta} \sin u \\
= \frac{du}{d\theta} \frac{d}{du} \sin u \\
= \left( 1 + \frac{1}{\theta} \right) \cos u \\
= \left( 1 + \frac{1}{\theta} \right) \cos(\theta + \ln \theta)
\]
2. (4 points) Given the equality \( x \sin y = 1 - y^2 \), find a formula for \( \frac{dy}{dx} \) by implicit differentiation.

Differentiating both sides of the equation with respect to \( x \):

\[
\frac{d}{dx}(x \sin y) = \frac{d}{dx}(1 - y^2)
\]

\[
\left( \frac{d}{dx}x \right) \sin y + x \frac{d}{dx} \sin y = - \frac{d}{dx} y^2
\]

\[
\sin y + x \frac{dy}{dx} \cos y = -\frac{dy}{dx} 2y
\]

\[
x \frac{dy}{dx} \cos y + 2y \frac{dy}{dx} = -\sin x
\]

\[
(x \cos y + 2y) \frac{dy}{dx} = -\sin x
\]

\[
\frac{dy}{dx} = \frac{-\sin x}{x \cos y + 2y}
\]

3. (4 points) An air compressor which delivers 10 cubic feet per minute is being used to inflate a spherical balloon of volume \( V \) and radius \( r \) which is currently 8 feet in radius; we wish to know how quickly the radius is increasing. Calculate \( \frac{dr}{dt} \) below, using the facts: \( V = \frac{4}{3} \pi r^3 \), \( \frac{dV}{dt} = 10 \), \( r = 8 \) currently.

We differentiate both sides of the expression \( V = \frac{4}{3} \pi r^3 \) with respect to \( t \), and then algebraically isolate \( \frac{dr}{dt} \).

\[
\frac{dV}{dt} = \frac{d}{dt} \frac{4}{3} \pi r^3
\]

\[
\frac{dV}{dt} = \frac{dr}{dt} \frac{4}{3} \pi r^2 \cdot \frac{dr}{dt}
\]

\[
\frac{dV}{dt} = \frac{4}{3} \pi r^2 \frac{dr}{dt}
\]

\[
\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4 \pi r^2}
\]

and since \( \frac{dV}{dt} = 10 \) and the current value of \( r \) is 8, we may conclude that the current value of \( \frac{dr}{dt} \) is \( \frac{10}{4 \pi (8^2)} = \frac{10}{256 \pi} = \frac{5}{128 \pi} \).

4. (2 point bonus) If \( n \) is a positive integer, find a general formula for \( \frac{d^n}{dx^n}(x^{n-1} \ln x) \) on the back of this page.
We might look at some specific examples, deriving each example from its predecessor:

\[
\frac{d}{dx} \ln x = \frac{1}{x}
\]

\[
\frac{d^2}{dx^2} (x \ln x) = \frac{d}{dx} \left( \ln x + \frac{x}{x} \right) = \frac{d}{dx} (\ln x + 1) = 1 \cdot \frac{1}{x} = \frac{1}{x}
\]

\[
\frac{d^3}{dx^3} (x^2 \ln x) = \frac{d^2}{dx^2} \left( 2x \ln x + \frac{x^2}{x} \right) = \frac{d^2}{dx^2} (2x \ln x + x) = 2 \cdot \frac{1}{x} = \frac{2}{x}
\]

\[
\frac{d^4}{dx^4} (x^3 \ln x) = \frac{d^3}{dx^3} \left( 3x^2 \ln x + \frac{x^3}{x} \right) = \frac{d^3}{dx^3} (3x^2 \ln x + x^2) = 3 \cdot \frac{2}{x} = \frac{6}{x}
\]

\[
\frac{d^5}{dx^5} (x^4 \ln x) = \frac{d^4}{dx^4} \left( 4x^3 \ln x + \frac{x^4}{x} \right) = \frac{d^4}{dx^4} (3x^3 \ln x + x^3) = 4 \cdot \frac{6}{x} = \frac{24}{x}
\]

and the pattern that emerges is that after taking one of the \( n \) derivatives requested of \( x^{n-1} \ln x \), we are left with \( \frac{d^{n-1}}{dx^{n-1}} ((n-1)x^{n-2} \ln x + x^{n-2}) \). The second term of this sum can be easily seen to be zero — hit \( x^{n-2} \) with \( n-1 \) derivatives and it will reach zero — while the first term of this sum looks a lot like the original value we were looking at, but with all exponents decreased by 1. If we repeat this process \( n - 1 \) times, we would get that

\[
\frac{d^n}{dx^n} (x^{n-1} \ln x) = \frac{(n-1)(n-2)(n-3) \cdots 2 \cdot 1}{x}
\]