1. **(4 points)** *Approximate the two following values with well-chosen linear approximations:*

   - $\sqrt[3]{997}$.

     We consider the function $f(x) = \sqrt[3]{x} = x^{1/3}$, whose derivative is $f'(x) = \frac{1}{3}x^{-2/3}$. For x close to 1000 (as 997 is), we can use the linear approximation:

     $$f(x) \approx f(1000) + (x - 1000)f'(1000)$$

     Since $f(1000) = 10$ and $f'(1000) = \frac{1}{3\sqrt[3]{1000}^2} = \frac{1}{300}$, it follows that

     $$f(997) \approx 10 - 3 \cdot \frac{1}{300} = 10 - 0.01 = 9.99$$

     For purposes of comparison, the actual value of $\sqrt[3]{997}$ is around 9.98998998.

   - 3.04^4.

     We consider the function $f(x) = x^4$, whose derivative is $f'(x) = 4x^3$. For x close to 3 (as 3.04 is), we can use the linear approximation:

     $$f(x) \approx f(3) + (x - 3)f'(3)$$

     Since $f(3) = 81$ and $f'(3) = 4 \cdot 27 = 108$, it follows that

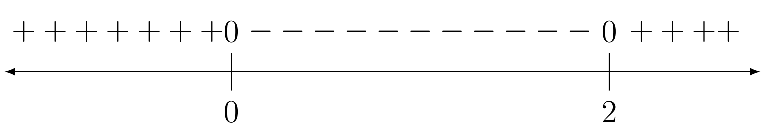
     $$f(3.04) \approx 81 + 0.04 \cdot 108 = 81 + 4.32 = 85.32$$

     For purposes of comparison, the actual value of 3.04^4 is around 85.40717.

2. **(10 points)** *Let $f(x) = 2 + 3x^2 - x^3$. Answer the following questions about this function.*

   (a) **(4 points)** *Where is it increasing? Where is it decreasing?*

   We calculate $f'(x) = 6x - 3x^2$; this is a polynomial so it is defined everywhere, and since $f'(x) = 3x(2 - x)$, it is evident that $f'(x) = 0$ at $x = 0$ and $x = 2$. Now we can probe in the three regions: when $x < 0$, when $0 < x < 2$, and when $x > 2$, with the three following respective calculations: $f'(-1) = 6(-1) - 3(-1)^2 = -9$, $f'(1) = 6(1) - 3(-1)^2 = 3$, and $f'(3) = 6(3) - 3(3)^2 = -9$, so the diagram of the sign of $f'(x)$ at various values is given by:



   so $f(x)$ is increasing where $f'(x)$ is positive, which is to say when $x < 0$ or $x > 2$. $f(x)$ is decreasing where $f'(x)$ is negative, which is to say when $0 < x < 2$.

   (b) **(3 points)** *Where are its local maxima and minima?*

   Since $x = 0$ is a transition from increase to decrease, it is a local maximum. Since $x = 2$ is a transition from decrease to increase, it is a local minimum.

   (c) **(3 points)** *Where is it concave up, where is it concave down, and what are its points of inflection?*

   We calculate $f''(x) = 6 - 3x$; this is a polynomial so it is defined everywhere; it is negative for $x > 2$ and positive for $x < 2$. Thus, $f(x)$ is concave up when $x < 2$ and concave down when $x > 2$; since $x = 2$ is the boundary between these two regions, it is a point of inflection.

3. **(6 points)** *Calculate the following limits:*

   (a) $\lim_{x\to 0} \frac{x-\sin x}{x^3 e^x}$.
   We may observe that $0 - \sin 0 = 0$ and $0^3 \cdot e^0 = 0$, so the argument of this limit is the indeterminate form $\frac{0}{0}$, justifying the use of L'Hôpital's rule (which will require the product rule in the denominator:
   $$\lim_{x\to 0} \frac{x-\sin x}{x^3 e^x} = \lim_{x\to 0} \frac{1-\cos x}{(x^3+3x^2)e^x}$$
   However, $1 - \cos 0 = 1 - 1 = 0$ and $(0^3 + 3 \cdot 0^2)e^0 = 0$, so this too is a $\frac{0}{0}$ form, requiring another use of L'Hôpital's rule (using the product rule again):
   $$\lim_{x\to 0} \frac{1-\cos x}{(x^3+3x^2)e^x} = \lim_{x\to 0} \frac{\sin x}{(x^3+6x^2+6x)e^x}$$
   and this will be again a $\frac{0}{0}$ form, requiring L'Hôpital's rule all over again to get
   $$\lim_{x\to 0} \frac{\sin x}{(x^3+6x^2+6x)e^x} = \lim_{x\to 0} \frac{\cos x}{(x^3+9x^2+12x+6)e^x} = \frac{1}{6}$$

   (b) $\lim_{t\to 3} \frac{t^2-2t-3}{t^2+2t+2}$.
   This is not a L'Hôpital's rule question! The numerator is 0, but the denominator is 17, and $\frac{0}{17} = 0$.

4. **(2 point bonus)** *Find an example of the equation of a continuous function with infinitely many local maxima, infinitely many global minima on $(-\infty, \infty)$, and only one global maximum.*

   A periodic function would be a good place to start: $\cos x$, for example, has infinitely many local maxima and minima, but they are all global. If we multiply by $\frac{1}{x^2+1}$, then only the middle peak and the two nearest troughs remain global. Finally, if we shift it downwards, all the troughs can be global. A suggested function is thus:
   $$f(x) = \frac{\cos x}{x^2+1} - \frac{x^2}{x^2+1}$$
   whose global maximum is attained at $x = 0$, and whose global minima are attained at $x = \ldots, -3\pi, -\pi, \pi, 3\pi, 5\pi, \ldots,$. This is not the only example.

> Before writing this essay, I examined 85 separate and distinct calculus books. I looked at all of their prefaces, all of their applications of maxima and minima, and all of their treatments of L'Hospital's Rule. By the way, I found five different spellings of L'Hospital. There were the two you would expect [L'Hôpital and L'Hospital –DJW], and Lhospital, as L'Hospital sometimes spelled his name. In addition, one author, not wanting to take chances, had it L'Hôspital, and one thought it was Le Hospital.
> —Underwood Dudley, Review of *Calculus with Analytic Geometry*