1. **(4 points)** *Using two steps of Newton's method with a starting value of $x_0 = 1$, produce an approximation x_2 to a zero of the function $f(x) = x^3 - x + 1$. Your answer need not be arithmetically simplified.*

   Let us note that $f'(x) = 3x^2 - 1$, and then:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^3 - 1 + 1}{3 \cdot 1^2 - 1} = \frac{1}{-}\frac{1}{2} = \frac{1}{2}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3 - \frac{1}{2} + 1}{3\left(\frac{1}{2}\right)^2 - 1} = \frac{1}{2} - \frac{\frac{5}{8}}{-\frac{1}{4}} = \frac{1}{2} + \frac{5}{2} = 3$$

2. **(8 points)** *An open-topped box with a square base is being built out of cardboard. You have 48 square feet of cardboard to use. What dimensions maximize the volume of the box?*

   Let our square base have its side length in feet denoted by x, and let the height of the box in feet be denoted by y. Thus the base is an $x \times x$ rectangle, with area x^2, while each sidewall is an $x \times y$ rectangle, with area xy. Since there are 4 sidewalls in total, the material required to build the box is $x^2 + 4xy$. Since we only have 48 square feet of material in total, we thus have the constraint that $x^2 + 4xy = 48$.

   The quantity we are seeking to maximize is the volume: since the box is $x \times x \times y$ in dimensions, its area will be $x \cdot x \cdot y = x^2 y$. We are thus trying to maximize $x^2 y$ subject to the constraint $x^2 + 4xy = 48$.

   We may write y in terms of x fairly easily (it is possible but difficult to write x in terms of y instead): $4xy = 48 - x^2$, so $y = \frac{48 - x^2}{4x}$. Now the expression for volume can be put entirely in terms of x:

$$V(x) = x^2 y = x^2 \frac{48 - x^2}{4x} = \frac{48x - x^3}{4}$$

   and we seek to maximize this on the interval $[0, \sqrt{48}]$. The lower edge of the interval has obvious physical necessity; the upper boundary arises fromt he fact that, since we use x^2 square feet of material for the base and 48 total, the largest x can be is such that $x^2 = 48$.

   Now we find the critical points of $V(x)$. $V'(x) = \frac{48 - 3x^2}{4}$, which is defiend everywhere, so the only critical points are where $48 - 3x^2 = 0$, or where $x^2 = 16$; thus $x = \pm 4$ are critical points of this function.

   Since -4 is not in the interval $[0, \sqrt{48}]$, we only have 3 candidates for the optimum: $x = 0$, $x = 4$, and $x = \sqrt{48}$. Unsurprisingly, $V(0) = V(\sqrt{48}) = 0$, so $x = 4$ is clearly the best of these three. If $x = 4$, then $y = \frac{48 - 4^2}{4 \cdot 4} = 2$, so the optimum dimensions for this box are $4' \times 4' \times 2'$.

3. **(8 points)** *Answer the following questions:*

   (a) **(4 points)** *What is the general antiderivative of the function $f(x) = \frac{4}{x^2} - 2\sqrt{x} + \frac{3}{1+x^2} + \csc^2 x$?*

   This is a sum of multiple terms; each is antidifferentiable in its own right. Note that $\frac{4}{x^2} = 4x^{-2}$ has antiderivative $\frac{4x^{-1}}{-1} = \frac{-4}{x}$, $2\sqrt{x} = 2x^{1/2}$ has antiderivative $\frac{2x^{3/2}}{3/2} = \frac{4}{3}x^{3/2}$, $\frac{3}{1+x^2}$ has antiderivative $3 \arctan x$, and $\csc^2 x$ has antiderivative $-\cot x$, so the entire expression has antiderivative

$$\frac{-4}{x} - \frac{4}{3}x^{3/2} + 3 \arctan x - \cot x + C$$

(b) **(4 points)** *If $h'(x) = 2e^x + 3\sin x$, and $h(0) = 6$, what is the formula for $h(x)$?*

We know that $h(x)$ is some antiderivative of $h'(x)$; the general antiderivative, calculated termwise, is $2e^x - 3\cos x + C$, so $h(x) = 2e^x - 3\cos x + C$ for some specific value of C. Since $h(0) = 6$, we may determine C specifically by evaluating $h(0)$:

$$6 = h(0) = 2e^0 - 3\cos 0 + C = 2 - 3 + C = C - 1$$

so $C = 7$ and thus $h(x) = 2e^x - 3\cos x + 7$.

4. **(2 point bonus)** *If $f'(x) = g(x)g'(x)$, and $f(0) = 0$, find and justify a formula for $f(x)$ in terms of $g(x)$.*

We might explore with a simple, familiar value of $g(x)$, as for instance if $g(x) = x^n$. In such a case, $f'(x) = x^n \cdot nx^{n-1} = nx^{2n-1}$, and using the general antiderivative we find that $f(x) = \frac{nx^{2n}}{2n} + C = \frac{1}{2}x^{2n} + C$. We note then that in this particular case $f(x) = \frac{1}{2}[g(x)]^2 + C$, and might wonder if that is true in general.

It is in fact easy to show, once we conjecture that $f(x) = \frac{1}{2}[g(x)]^2 + C$, that it does in fact satisfy the first criterion given, since $f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}[g(x)]^2 + C\right) = \frac{1}{2} \cdot 2g(x)g'(x)$. In order to satisfy the second criterion, we must choose C such that $f(0) = 0$; since $f(0) = \frac{1}{2}[g(0)]^2 + C$, it works to let $C = -\frac{[}{g}(0)]^2$. Thus the function $f(x)$ is given by the formula

$$f(x) = \frac{[g(x)]^2 - [g(0)]^2}{2}$$