

1. (16 points) Answer the following derivative-related questions.

(a) (8 points) If $y = \tan \sqrt{\sec x}$, find $\frac{dy}{dx}$.

This is an application of the chain rule in two stages, in which we shall let $u = \sqrt{\sec x} = \sqrt{v}$, and $v = \sec x$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \tan \sqrt{\sec x} \\ &= \frac{d}{dx} \tan u \\ &= \frac{du}{dx} \frac{d}{du} \tan u \\ &= \left(\frac{d}{dx} \sqrt{v} \right) \frac{d}{du} \tan u \\ &= \frac{dv}{dx} \left(\frac{d}{dv} \sqrt{v} \right) \frac{d}{du} \tan u \\ &= (\sec x \tan x) \cdot \frac{1}{2\sqrt{v}} \cdot \sec^2 u \\ &= \sec x \tan x \cdot \frac{1}{2\sqrt{\sec x}} \cdot \sec^2 \sqrt{\sec x} \end{aligned}$$

(b) (4 points) Find $\frac{d}{dx} ((e^x + 2x) \arcsin x)$.

This is an application of the product rule:

$$\begin{aligned} \frac{d}{dx} ((e^x + 2x) \arcsin x) &= \frac{d}{dx} (e^x + 2x) \arcsin x + (e^x + 2x) \frac{d}{dx} \arcsin x \\ &= (e^x + 2) \arcsin x + (e^x + 2x) \cdot \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

(c) (4 points) If $g(t) = \frac{\ln t}{\sqrt[3]{t}}$, find $g'(t)$.

This is an application of the quotient rule:

$$\begin{aligned} g'(t) &= \frac{d}{dt} \frac{\ln t}{\sqrt[3]{t}} \\ &= \frac{\sqrt[3]{t} \left(\frac{d}{dt} \ln t \right) - \ln t \frac{d}{dt} t^{1/3}}{(\sqrt[3]{t})^2} \\ &= \frac{\sqrt[3]{t} \cdot \frac{1}{t} - \ln t \cdot \frac{1}{3} t^{-2/3}}{(\sqrt[3]{t})^2} \end{aligned}$$

This problem could alternatively be solved as a product-rule invocation on $g'(t) = \ln t \cdot t^{-1/3}$.

2. (12 points) Calculate $\frac{d}{dx} \cos(x^2 e^x)$.

Let $u = x^2e^x$; then this is a straightforward chain rule invocation, although with a product rule embedded within it:

$$\begin{aligned}\frac{d}{dx} \cos(x^2e^x) &= \frac{d}{dx} \cos(u) \\ &= \frac{du}{dx} \frac{d}{du} \cos(u) \\ &= \frac{d}{dx} (x^2e^x) (-\sin u) \\ &= (2xe^x + x^2e^x) (-\sin u) \\ &= (2xe^x + x^2e^x) (-\sin(x^2e^x))\end{aligned}$$

3. **(10 points)** Find an equation of the tangent line to the curve $y = e^x(x^2 - 3x + 1)$ at $(0, 1)$.

Using the product rule,

$$\frac{dy}{dx} = e^x(x^2 - 3x + 1) + e^x(2x - 3)$$

and specifically when $x = 0$, $\frac{dy}{dx} = e^0 \cdot 1 + e^0(-3) = -2$, so the desired line has slope -2 , and thus equation $y = -2x + b$. Since it passes through the point $(0, 1)$ we may specifically determine that $1 = -2 \cdot 0 + b$, and thus that $b = 1$, for an equation $y = -2x + 1$ of the tangent line.

4. **(15 points)** The cissoid of Diocles is a curve satisfying the equation $x(x^2 + y^2) = 4y^2$.

- (a) **(12 points)** Find a formula for $\frac{dy}{dx}$ on this curve.

Taking the derivative of each side, we get

$$\begin{aligned}\frac{d}{dx} (x(x^2 + y^2)) &= \frac{d}{dx} (4y^2) \\ (x^2 + y^2) + x \frac{d}{dx} (x^2 + y^2) &= \frac{dy}{dx} \frac{d}{dy} (4y^2) \\ (x^2 + y^2) + 2x^2 + x \frac{d}{dx} y^2 &= \frac{dy}{dx} 8y \\ 3x^2 + y^2 + x \frac{dy}{dx} \frac{d}{dy} y^2 &= \frac{dy}{dx} 8y \\ 3x^2 + y^2 + 2xy \frac{dy}{dx} &= 8y \frac{dy}{dx} \\ 2xy \frac{dy}{dx} - 8y \frac{dy}{dx} &= -3x^2 - y^2 \\ (2xy - 8y) \frac{dy}{dx} &= -3x^2 - y^2 \\ \frac{dy}{dx} &= \frac{-3x^2 - y^2}{2xy - 8y} = \frac{3x^2 + y^2}{8y - 2xy}\end{aligned}$$

- (b) **(3 points)** Find the equation of the tangent line to the curve at $(2, -2)$.

At the specific values $x = 2$, $y = -2$, we see from the above that

$$\frac{dy}{dx} = \frac{3 \cdot 2^2 + (-2)^2}{8(-2) - 2 \cdot 2(-2)} = \frac{16}{-8} = -2$$

Thus we have a line of the form $y = -2x + b$; plugging in $(2, -2)$ we can solve for b :

$$\begin{aligned} -2 &= -2 \cdot 2 + b \\ 2 &= b \end{aligned}$$

so $y = -2x + 2$ is the tangent line.

5. **(9 points)** Find the absolute maxima and minima of the function $f(x) = x^3 - 2x^2 - 4x + 3$ on the interval $[0, 3]$.

We know that

$$f'(x) = 3x^2 - 4x - 4$$

We then seek out critical points. $f'(x)$ is a polynomial, so points where $f'(x)$ does not exist are not under consideration. We also look for values where $f'(x) = 0$. This occurs when $3x^2 - 4x - 4 = 0$, which, either by factorization or by the quadratic formula can be found to be the case when $x = 2$ or $x = -\frac{2}{3}$. $x = -\frac{2}{3}$ is outside our interval, so the only critical point within the interval is $x = 2$.

Thus, the only candidates for minimum and maximum are the critical point $x = 2$ and the endpoints $x = 0$ and $x = 3$. $f(0) = 0^3 - 2 \cdot 0^2 - 4 \cdot 0 + 3 = 3$, $f(2) = 2^3 - 2 \cdot 2^2 - 4 \cdot 2 + 3 = -5$, and $f(3) = 3^3 - 2 \cdot 3^2 - 4 \cdot 3 + 3 = 0$. Of these, we may thus say with certainty that the highest value (the absolute maximum) is at $x = 0$, and the lowest value (the absolute minimum) at $x = 2$.

6. **(16 points)** Amy is standing motionless 50 meters east of a north-south road with a radar gun, while Bob, who is 120 meters to the north, is driving south. The radar gun reports how quickly the distance between Amy and Bob is changing (which may not be Bob's actual speed).

Our setup for this problem involves a right triangle between Bob's car, the place on the road parallel to Amy, and Amy's position well off the road. The east-west leg between Amy and the road is a constant length of 50, while the length of the north-south leg and the length of the hypotenuse are changing. We may denote these respectively by y and s , and note that, by the Pythagorean Theorem, they have the relationship $y^2 + 50^2 = s^2$. We also know that y is currently 120, and can compute in addition that s is currently

$$\sqrt{50^2 + y^2} = \sqrt{2500 + 14400} = \sqrt{16900} = 130$$

- (a) **(12 points)** If Bob is driving south at 30 meters per second, what will the radar report as the rate of change of the distance between Bob and Amy?

The information given to us here is that $\frac{dy}{dt} = -30$ (since y is decreasing at a rate of 30 meters per second), and that we wish to find $\frac{ds}{dt}$. Using the relationship described in the

genral problem, we differentiate both sides with respect to t :

$$\begin{aligned} y^2 + 50^2 &= s^2 \\ \frac{d}{dt}(y^2 + 50^2) &= \frac{d}{dt}s^2 \\ \frac{dy}{dt} \frac{d}{dy}(y^2 + 50^2) &= \frac{ds}{dt} \frac{d}{ds}s^2 \\ \frac{dy}{dt} 2y &= \frac{ds}{dt} 2s \\ \frac{\frac{dy}{dt}y}{s} &= \frac{ds}{dt} \end{aligned}$$

and thus $\frac{ds}{dt} = \frac{-30 \cdot 120}{130} = \frac{-36}{13}$.

- (b) **(4 points)** *Conversely, if the radar reported a change-rate of 25 meters per second, what would Bob's actual speed be?*

In this case we are given $\frac{ds}{dt} = \pm 25$, and asked to find $\frac{dy}{dt}$. We may start with the second-to last line of the derivation seen in the previous section:

$$\frac{dy}{dt} 2y = \frac{ds}{dt} 2s$$

and solve for $\frac{dy}{dt} = \frac{\frac{ds}{dt}s}{y} = \frac{\pm 25 \cdot 130}{120} = \pm \frac{325}{12}$.

7. **(9 points)** *Estimate the following values using appropriate linear approximations.*

- (a) **(4 points)** $(-1.993)^4$.

We consider the function $f(x) = x^4$, whose derivative is $f'(x) = 4x^3$. For x close to -2 (as -1.993 is), we can use the linear approximation:

$$f(x) \approx f(-2) + (x + 2)f'(-2)$$

Since $f(-2) = 16$ and $f'(-2) = -32$, it follows that

$$f(-1.993) \approx 16 + 0.007(-32) = 16 - 0.224 = 15.776$$

For purposes of comparison, the actual value of $(-1.993)^4$ is around 15.777173.

- (b) **(5 points)** $\sqrt{25.07}$.

We consider the function $f(x) = \sqrt{x}$, whose derivative is $\frac{1}{2\sqrt{x}}$. For x close to 25 (as 25.07 is), we can use the linear approximation:

$$f(x) \approx f(25) + (x - 25)f'(25)$$

Since $f(25) = 5$ and $f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$, it follows that

$$f(25.07) \approx 5 + (0.07) \cdot \frac{1}{10} = 5.007$$

For purposes of comparison, the actual value of $\sqrt{25.07}$ is around 5.0069951.

8. **(13 points)** If $f(x) = \frac{\arctan 3x}{\sqrt{x^4+2}}$, then find $f'(x)$.

We shall need the quotient rule immediately, and will eventually need recourse to the chain rule; preemptively we shall define $u = 3x$ and $v = x^4 + 2$, which we will need later. Then:

$$\begin{aligned}
 f'(x) &= \frac{d}{dx} \frac{\arctan u}{\sqrt{v}} \\
 &= \frac{\sqrt{v} \left(\frac{d}{dx} \arctan u \right) - \arctan u \frac{d}{dx} \sqrt{v}}{(\sqrt{v})^2} \\
 &= \frac{\sqrt{v} \left(\frac{du}{dx} \frac{d}{du} \arctan u \right) - \arctan u \frac{dv}{dx} \frac{d}{dv} \sqrt{v}}{v} \\
 &= \frac{\sqrt{v} \left(\frac{d}{dx} 3x \right) \cdot \frac{1}{1+u^2} - \arctan u \left(\frac{d}{dx} x^4 + 2 \right) \frac{1}{2\sqrt{v}}}{v} \\
 &= \frac{\sqrt{x^4+2} \cdot 3 \cdot \frac{1}{1+(3x)^2} - \arctan(3x) \cdot 4x^3 \cdot \frac{1}{2\sqrt{x^4+2}}}{x^4+2} \\
 &= \frac{\frac{3\sqrt{x^4+2}}{1+9x^2} - \frac{4x^3 \arctan(3x)}{2\sqrt{x^4+2}}}{x^4+2}
 \end{aligned}$$

The last line of the above calculation is cleanup and is optional.

9. **(6 point bonus)** Currently Yvette is 10 miles north of the Library of Babel, walking south at 3mph, while Zachary is 1 mile east of the Library, walking east at 5mph. How soon will it be the case that the distance between them is (if only momentarily) unchanging?

This can be solved either purely as a derivative calculation, or as a related rates problem. The key observation is that we are interested in various properties at times *other* than the present, so it would be erroneous to work under the presumption that Yvette and Zachary's distances from the library are specifically 10 and 1 mile respectively; instead, in t hours, their distances will be $10 - 3t$ and $1 + 5t$ respectively.

To solve this purely as a derivative, one could then compute the distance between them at time t to be

$$s = \sqrt{(10 - 3t)^2 + (1 + 5t)^2} = \sqrt{34t^2 - 50t + 101}$$

and then using the chain rule,

$$\frac{ds}{dt} = \frac{68t - 50}{2\sqrt{34t^2 - 50t + 101}}$$

and since we want to know *when* $\frac{ds}{dt}$ is zero, we set this expression to zero and solve for t , getting $t = \frac{50}{68} = \frac{25}{34}$.

Alternatively, we could let Yvette's and Zachary's distances from the library be given by the variables $y = 10 - 3t$ and $z = 1 + 5t$, and by the further known derivatives $\frac{dy}{dt} = -3$ and

$\frac{dz}{dt} = 5$. We know $y^2 + z^2 = s^2$, so differentiating both sides with respect to t :

$$\begin{aligned}\frac{d}{dt}(y^2 + z^2) &= \frac{d}{dt}s^2 \\ \frac{dy}{dt} \frac{d}{dy}y^2 + \frac{dz}{dt} \frac{d}{dz}z^2 &= \frac{ds}{dt} \frac{d}{ds}s^2 \\ \frac{dy}{dt}2y + \frac{dz}{dt}2z &= \frac{ds}{dt}2s \\ 2 \cdot (-3)(10 - 3t) + 2 \cdot 5(1 + 5t) &= \frac{ds}{dt}2s\end{aligned}$$

and since we are interested in when $\frac{ds}{dt} = 0$, this equation can be simplified to $-50 + 68t = 0$, with solution $t = \frac{25}{34}$.