1. **(14 points)** *Answer the following questions about approximation:*

   (a) **(7 points)** *Choose $x_0 = 2$ to be an initial approximation of $\sqrt[4]{20}$. Use one step of Newton's method on an appropriately chosen polynomial function to develop x_1, a better rational approximation of $\sqrt[4]{20}$; also give an arithmetic expression (which need not be simplified) for the better approximation x_2 arising from a second step of Newton's method.*

   We want an easy-to-evaluate polynomial of which $\sqrt[4]{20}$ is a zero, in order for Newton's method to help us approximate it; the obvious choice is $f(x) = x^4 - 20$. Note that $f'(x) = 4x^3$. Then,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2^4 - 20}{4 \cdot 2^3} = 2 - \frac{-4}{32} = 2 + \frac{1}{8} = \frac{17}{8}$$

   And we follow up with the further improvement

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{17}{8} - \frac{\left(\frac{17}{8}\right)^4 - 20}{4 \cdot \left(\frac{17}{8}\right)^3}$$

   This last expression is far too complicated to easily simplify by hand, but it is actually $\frac{332483}{157216}$, which is within 0.00005 of the correct value of $\sqrt[4]{20}$.

   (b) **(7 points)** *Starting with an initial value of 1, use two iterations of Newton's method to approximate a zero of $f(x) = x^5 - x^2 - 2x + 5$. Your answer need not be arithmetically simplified.*

   Let us start by observing that $f'(x) = 5x^4 - 2x - 2$. Using Newton's method once:

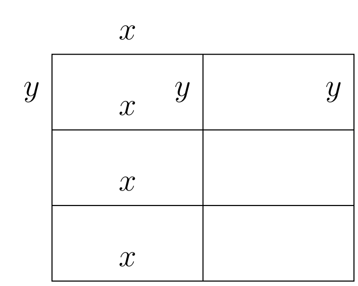
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^5 - 1^2 - 2 \cdot 1 + 5}{5 \cdot 1^4 - 2 \cdot 1 - 2} = 1 - \frac{3}{1} = -2$$

   And using it again:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -2 - \frac{(-2)^5 - (-2)^2 - 2(-2) + 5}{5 \cdot (-2)^4 - 2 \cdot (-2) - 2} = -2 - \frac{-27}{82} = \frac{-137}{82}$$

   This isn't terribly close to the correct result of -1.42236, but a few more iterations would get it closer.

2. **(20 points)** *You wish to fence a 4800-square-foot rectangular field and subdivide it into six plots by placing two internal fences parallel to one pair of sides, and one internal fence parallel to the other pair of sides. What dimensions for the field will use the least quantity of fencing?*



The above drawing is a representation of the scenario described; we assign the two dimensions of the field the labels of x and y; note that each of the seven fences then have lengths of x or y.

Our goal is to minimize the total quantity of fencing used, subject to the condition that the area of the field is 4800 square feet. Since the total length of fencing can be seen to be $4x+3y$, and the area of the field is xy, these correspond to the problem of minimizing $4x+3y$ subject to the constraint that $xy = 4800$. We may re-express this constraint as $y = \frac{4800}{x}$, so that the expression we seek to minimize is, in a single variable, $f(x) = 4x + 3 \cdot \frac{4800}{x} = 4x + \frac{14400}{x}$. The interval of acceptable values on x is $[0, \infty)$, since our field cannot have negative width but has no bound on its upper length.

Since $f'(x) = 4 - \frac{14400}{x^2}$, we see that $f'(x)$ will have 3 critical points: one when $x = 0$, where $f'(x)$ is undefined, and two at the solutions to $4 - \frac{14400}{x^2} = 0$, which occur at $x = \pm\sqrt{3600} = \pm 60$. Our options for maximizing choices of x are thus the 3 critical points 0, -60, and 60, together with the interval endpoints 0 and the limiting behavior as $x \to \infty$. -60 is outside the interval and may be rejected out of hand. Evaluating at the other three points, we see that $f(x)$ does not exist at $x = 0$, but that $\lim_{x\to 0^+} f(x) = +\infty$; $\lim_{x\to+\infty} f(x) = +\infty$, and $f(60) = 4 \cdot 60 + \frac{14400}{60}$. This last value need not be completely calculated; it is finite, and thus a better minimum than the other two prospects. Thus, our area-minimizing choice of x is 60, which has an associated value of $y = \frac{4800}{60} = 80$, so our optimal dimensions are 60×80.

3. **(24 points)** *Answer the following questions related to the shape of the graph of* $g(x) = e^x(x-4)$.

   (a) **(4 points)** *When is $g(x)$ equal to zero? What is its y-intercept? Label which is which.*

   The y-intercept of $g(x)$ is given by $g(0) = e^0(0-4) = 1(-4) = -4$. The zeroes of $g(x)$ are given by the solutions to the equation $e^x(x-4) = 0$, which, since e^x is always positive, will occure only when $x = 4$.

   (b) **(7 points)** *Where is it increasing? Where is it decreasing? Label which is which.*

   $f'(x) = e^x(x-4) + e^x \cdot 1 = e^x(x-3)$; since e^x is always positive, the sign of this quantity is dictated by the sign of $x - 3$; thus, $f'(x)$ is positive if $x > 3$ and negative if $x < 3$, so $f(x)$ is increasing for $x > 3$ and decreasing for $x < 3$.

   (c) **(6 points)** *What are its critical points, and is each a local maximum, a local minimum, or neither?*

   From the factorization above, it is clear that $f(x) = 0$ when $x = 3$. Since $f(x)$ decreases up to 3 and increases from it, $x = 3$ is a local minimum. This result can also be obtained via the second derivative test, but doing so when you already have the signs of the first derivative at hand is not terribly necessary.

   (d) **(7 points)** *Where is it concave up? Where is it concave down? Label which is which. Where, if anywhere, are its points of inflection?*

   $f''(x) = e^x(x-3) + e^x = e^x(x-2)$, so $f''(x) > 0$ if $x > 2$, and $f''(x) < 0$ if $x < 2$. Thus, $f(x)$ is concave up when $x > 2$, concave down when $x < 2$, and has a point of inflection at $x = 2$.

4. **(22 points)** *Evaluate the following limits; if they cannot be evaluated, show why not.*

(a) $\lim_{x\to 0} \frac{x^2}{1-\cos x}$.

Note that $0^2 = 0$ and $1 - \cos 0 = 1 - 1 = 0$, so this is a $\frac{0}{0}$ indeterminate form. Using L'Hôpital's rule:

$$\lim_{x\to 0} \frac{x^2}{1-\cos x} = \lim_{x\to 0} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}(1-\cos x)} = \lim_{x\to 0} \frac{2x}{\sin x}$$

But this is still a $\frac{0}{0}$ form, since $2 \cdot 0 = 0$ and $\sin 0 = 0$, so we use L'Hôpital's rule again:

$$\lim_{x\to 0} \frac{2x}{\sin x} = \lim_{x\to 0} \frac{\frac{d}{dx}2x}{\frac{d}{dx}\sin x} = \lim_{x\to 0} \frac{2}{\cos x}$$

which can be evaluated to be $\frac{2}{\cos 1} = 2$.

(b) $\lim_{\theta\to 0} \frac{\tan\theta}{\cos\theta}$.

This one can be solved by direct evaluation: $\lim_{\theta\to 0} \frac{\tan\theta}{\cos\theta} = \frac{\tan 0}{\cos 0} = \frac{0}{1} = 0$.

(c) $\lim_{t\to 0} \frac{e^{6t}-e^{2t}}{t^2}$.

Note that $e^0 - e^0 = 0$ and $0^2 = 0$, so this is a $\frac{0}{0}$ indeterminate form. Using L'Hôpital's rule:

$$\lim_{t\to 0} \frac{e^{6t}-e^{2t}}{t^2} = \lim_{t\to 0} \frac{\frac{d}{dt}\left(e^{6t}-e^{2t}\right)}{\frac{d}{dt}t^2} = \lim_{t\to 0} \frac{6e^{6t}-2e^{2t}}{2t}$$

which is no logner an indeterminate form, since $6e^0 - 2e^0 = 4$. However,t he denominator is still zero, so we have an unevaluatable limit.

(d) $\lim_{x\to +\infty} \frac{\ln(x+3)}{x^2}$.

As x grows without bound, so do both $\ln(x+3)$ and x^2, so this is a $\frac{\infty}{\infty}$ indeterminate form. Applying L'Hôpital's rule to this form gives

$$\lim_{x\to +\infty} \frac{\ln(x+3)}{x^2} = \lim_{x\to +\infty} \frac{\frac{1}{x+3}}{2x} = \lim_{x\to +\infty} \frac{1}{2x(x+3)}$$

in which the denominator grows without bound while the numerator, being a constant, does not. Thus, this expression approaches zero as x increases, so the limit is equal to zero.

(e) $\lim_{u\to 1} \frac{u^4-1}{\ln u}$.

Note that $1^4 - 1 = 0$ and $\ln 1 = 0$, so this is a $\frac{0}{0}$ indeterminate form. Using L'Hôpital's rule:

$$\lim_{u\to 1} \frac{u^4-1}{\ln u} = \lim_{u\to 1} \frac{\frac{d}{du}(u^4-1)}{\frac{d}{du}\ln u} = \lim_{u\to 1} \frac{4u^3}{\frac{1}{u}}$$

which can be evaluated to be $\frac{4\cdot 1^3}{\frac{1}{1}} = 4$.

5. **(20 points)** *Answer the following questions:*

(a) **(7 points)** *Find* $g(t)$ *given that* $g'(t) = t - \frac{1}{t^3}$ *and* $g(1) = 6$.

We know that $g(t)$ is an antiderivative of $g'(t)$; the general antiderivative, worked term-by term, can be seen to be $\frac{t^2}{2} - \frac{t^{-2}}{-2} + C$; thus we know that $g(t) = \frac{t^2+t^{-2}}{2} + C$ for *some* vale of C. Plugging in the known value $g(1) = 6$, we can solve for C:

$$6 = \frac{1+1}{2} + C$$
$$5 = C$$

so the specific formula for $g(t)$ is $\frac{t^2+t^{-2}}{2} + 5$.

(b) **(6 points)** *Determine a region whose area is* $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{7}{n}\right) \sqrt{1 + \frac{7i}{n}}$.

The area of a region under a curve $f(x)$ from $x = a$ to $x = b$ is known to be

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \frac{b-a}{n} i\right),$$

which bears a similarity except in details to the expression given. We must therefore find expressions to stand in for a, b, and $f(x)$ to make these expressions equivalent. Taking the most naïve decomposition of these equivalent expressions, we find that

$$\frac{b-a}{n} = \frac{7}{n}$$
$$f\left(a + \frac{b-a}{n} i\right) = \sqrt{1 + \frac{7}{n} i}$$

The first equation clearly establishes that $b - a = 7$; substituting this knowledge into the second equation, we find that our correspondences become:

$$b - a = 7$$
$$f\left(a + \frac{7}{n} i\right) =$$
$$sqrt 1 + \frac{7}{n} i$$

which lends itself to the obvious intepretation $f(x) = \sqrt{x}$ and $a = 1$ (other interpretations are possible, and will give rise to slightly different but equally correct answers). Then, since $b - a = 7$, $b = 7 + a = 8$. Thus, the expression we were given is the area-under-a-curve formula with $a = 1$, $b = 8$, and $f(x) = \sqrt{x}$, so this expression is the area of the region under the curve $y = \sqrt{x}$ between $x = 1$ and $x = 8$, which might be written as $\int_1^8 \sqrt{x} dx$.

(c) **(7 points)** *Find the general antiderivative of* $h(\theta) = 2\cos\theta + \sec^2\theta - \frac{4}{\theta} + \frac{6}{\sqrt[3]{\theta}}$.

We intepret this expression as $h(\theta) = 2\cos\theta + \sec^2\theta - 4\theta^{-1} + 6\theta^{-1/3}$. Using known antiderivative rules, we know that antiderivatives for $\cos\theta$, $\sec^2\theta$, θ^{-1}, and $\theta^{-1/3}$ are $\sin\theta$, $\tan\theta$, $\ln|\theta|$, and $\frac{\theta^{2/3}}{2/3}$ respectively, so the general antiderivative of $h(\theta)$ is

$$3\sin\theta + \tan\theta - 4\ln|\theta| + \frac{6\theta^{2/3}}{2/3} + C = 3\sin\theta + \tan\theta - 4\ln|\theta| + 9\theta^{2/3} + C$$