1. **(6 points)** *Compute the following infinite limits, or, if the limit is uncomputable, briefly describe (in symbols or words) the relevant long-term behavior of the function.*

   (a) **(2 points)** $\lim\limits_{x\to-\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5}$.

   Using the highest-order-term technique, we can be certain that for values of x of very large magnitude (large negative magnitude, in this case), $\frac{4x^3+6x^2-2}{2x^3-4x+5} \approx \frac{4x^3}{2x^3} = 2$. Thus, $\lim\limits_{x\to-\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5} = 2$.

   Alternatively, using the technique of division by the highest degree in the denominator,

$$\lim_{x\to-\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5} = \lim_{x\to-\infty} \frac{\frac{4x^3+6x^2-2}{x^3}}{\frac{2x^3-4x+5}{x^3}} = \lim_{x\to-\infty} \frac{4+\frac{6}{x}-\frac{2}{x^3}}{2-\frac{4}{x^2}+\frac{5}{x^3}} = \frac{4+0-0}{2-0+0} = 2$$

   (b) **(2 points)** $\lim\limits_{t\to+\infty} \frac{(t-2)(t+5)}{t^2(2t-6)}$.

   Using the highest-order-term technique, we can be certain that for values of t of very large magnitude (large positive magnitude, in this case), $\frac{(t-2)(t+5)}{t^2(2t-6)} = \frac{t^2+3t-10}{2t^3-6t^2} \approx \frac{t^2}{2t^3} = \frac{1}{2t}$. Thus, $\lim\limits_{t\to+\infty} \frac{(t-2)(t+5)}{t^2(2t-6)} = \lim\limits_{t\to+\infty} \frac{1}{2t} = 0$.

   Alternatively, using the technique of division by the highest degree in the denominator,

$$\lim_{t\to+\infty} \frac{(t-2)(t+5)}{t^2(2t-6)} = \lim_{t\to+\infty} \frac{\frac{(t-2)(t+5)}{t^3}}{\frac{t^2(2t-6)}{t^3}} = \lim_{t\to+\infty} \frac{\left(1-\frac{2}{t}\right)\left(\frac{1}{t}+\frac{5}{t^2}\right)}{2-\frac{6}{t}} = \frac{(1-0)(0+0)}{2-0} = 0$$

   (c) **(2 points)** $\lim\limits_{r\to-\infty} \frac{2-r^3}{r^2+2r+1}$.

   Using the highest-order-term technique, we can be certain that for values of r of very large magnitude (large negative magnitude, in this case), $\frac{2-r^3}{r^2+2r+1} \approx \frac{-r^3}{r^2} = -r$. Thus, $\lim\limits_{r\to-\infty} \frac{2-r^3}{r^2+2r+1} = \lim\limits_{r\to-\infty} -r$. Note that this limit does not actually exist but follows an easy to describe behavior: as r becomes a very large-magnitude negative number, $-r$, and by inference $\frac{2-r^3}{r^2+2r+1}$ will necessarily be a positive nubmer of equally enormous magnitude; thus, we could say that $\frac{2-r^3}{r^2+2r+1}$ increases without bound, or if we wished to write it purely symbolically, $\lim_{r\to-\infty} \frac{2-r^3}{r^2+2r+1} = +\infty$.

   Alternatively, using the technique of division by the highest degree in the denominator,

$$\lim_{r\to-\infty} \frac{2-r^3}{r^2+2r+1} = \lim_{r\to-\infty} \frac{\frac{2-r^3}{r^2}}{\frac{r^2+2r+1}{r^2}} = \lim_{r\to+\infty} \frac{\frac{2}{r^2}-r}{1+\frac{2}{r}+\frac{1}{r^2}}$$

   and as $r \to -\infty$, we see that several of these terms dwindle towards zero, leavign us with $\frac{0-r}{1+0+0}$; we then proceed to analyze its behavior as in the other method.

2. **(4 points)** *Using the difference quotient*, find the derivative of the function $f(x) = 4x^2 - 5x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[4(x+h)^2-5(x+h)]-(4x^2-5x)}{h} \\ &= \lim_{h\to 0} \frac{(4x^2+8xh+4h^2-5x-5h)-(4x^2-5x)}{h} \\ &= \lim_{h\to 0} \frac{8xh+4h^2-5h}{h} \\ &= \lim_{h\to 0} 8x+4h-5 \\ &= 8x-5 \end{aligned}$$

3. **(4 points)** *If* $g(t) = 6t^3 - \frac{1}{\sqrt{t}} + 6e^t$*, compute* $g''(t)$*.*

   Let us start by recasting $g(t)$ in a form which is amenable to differentiation; i.e. rewrite the square root specifically as a power function:

   $$g(t) = 6t^3 - t^{-1/2} + 6e^t$$

   and now we can take the derivative of this expression to get $g'(t)$:

   $$g'(t) = \frac{d}{dt}g(t) = \frac{d}{dt}\left(6t^3 - t^{-1/2} + 6e^t\right) = 18t^2 + \frac{1}{2}t^{-3/2} + 6e^t$$

   and we get the second derivative by once again differentiating:

   $$g''(t) = \frac{d}{dt}g'(t) = \frac{d}{dt}\left(18t^2 + \frac{1}{2}t^{-3/2} + 6e^t\right) = 36t^2 - \frac{3}{4}t^{-5/2} + 6e^t$$

   Note that the middle term could alternatively be written, if desired, as $-\frac{3}{4\sqrt{t^5}}$.

4. **(6 points)** *Perform the following computations; you do not need to algebraically simplify fully differentiated expressions.*

   (a) **(3 points)** *Determine* $\frac{d}{ds}\frac{s^3-3s}{5s^2-e^s}$*.*

   We invoke the quotient rule:

   $$\begin{aligned} \frac{d}{ds}\frac{s^3-3s}{5s^2-e^s} &= \frac{(5s^2-e^s)\frac{d}{ds}(s^3-3s) - (s^3-3s)\frac{d}{ds}(5s^2-e^s)}{(5s^2-e^s)^2} \\ &= \frac{(5s^2-e^s)(3s^2-3) - (s^3-3s)(10s^2-e^s)}{(5s^2-e^s)^2} \end{aligned}$$

   And further simplification is, as mentioned, not necessary.

   (b) **(3 points)** *If* $y = (t^e + e^t)\left(3\sqrt{t} - \frac{1}{t^3}\right)$*, find* $\frac{dy}{dt}$*.*

   Noting that $\sqrt{t} = t^{1/2}$ and $\frac{1}{t^3} = t^{-3}$, we invoke the quotient rule:

   $$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}\left[(t^e+e^t)(3\sqrt{t} - \frac{1}{t^3})\right] \\ &= \left[\frac{d}{dt}(t^e+e^t)\right](3\sqrt{t} - \frac{1}{t^3}) + (t^e+e^t)\frac{d}{dt}(3t^{1/2} - t^{-3}) \\ &= (et^{e-1}+e^t)(3\sqrt{t} - \frac{1}{t^3}) + (t^e+e^t)(\frac{3}{2}t^{-1/2} + 3t^{-4}) \end{aligned}$$

And further simplification is, as mentioned, not necessary.

5. **(2 point bonus)** *If $f(x) = x^2e^x$, find (with justification) a general formula for $f^{(n)}(x)$ on the back of this page.*

We might pre-emptively note, by using the product rule, that $\frac{d}{dx}(x^2e^x) = x^2e^x + 2xe^x$ and that $\frac{d}{dx}(xe^x) = xe^x + e^x$; additionally, we see that $\frac{d}{dx}e^x = e^x$ as we already knew. We thus reasonably expect that repeated differentiation of z^2e^x will spawn terms of the form x^2e^x, xe^x, and e^x, since only terms of those forms result from taking the derivatives of terms of those forms.

If we perform a few simple derivatives, we can see patterns emerging.

$$\begin{aligned}
f(x) &= x^2e^x & &= x^2e^x \\
f'(x) &= x^2e^x + 2xe^x & &= x^2e^x + 2xe^x \\
f''(x) &= x^2e^x + 2xe^x + 2xe^x + 2e^x & &= x^2e^x + 4xe^x + 2e^x \\
f'''(x) &= x^2e^x + 2xe^x + 4xe^x + 4e^x + 2e^x & &= x^2e^x + 6xe^x + (2+4)e^x \\
f^{(4)}(x) &= x^2e^x + 2xe^x + 6xe^x + 6e^x + (2+4)e^x & &= x^2e^x + 8xe^x + (2+4+6)e^x
\end{aligned}$$

and we could form a conjecture (which is not too hard to prove, although it would be easiest to do using the principle of induction, which is not covered in this class), that

$$f^{(n)}(x) = x^2e^x + nxe^x + (2 + 4 + \cdots + 2(n-1))e^x$$

and, using rules for simplifying arithmetic series, the last term can actually be folded into a closed form to yield:

$$f^{(n)}(x) = x^2e^x + nxe^x + n(n-1)e^x$$