

1. (12 points) Compute the following derivatives:

(a) (4 points) Given  $y = x^2 e^{\sec x}$ , find  $\frac{dy}{dx}$ .

This is a product rule stacked with a chain rule (since  $e^{\sec x}$  is a composition). Let us pre-emptively assign  $u = \sec x$ , so that

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} (x^2 e^{\sec x}) \\ &= \left( \frac{d}{dx} x^2 \right) e^{\sec x} + x^2 \frac{d}{dx} e^u \\ &= 2x e^{\sec x} + x^2 \frac{du}{dx} \frac{d}{du} e^u \\ &= 2x e^{\sec x} + x^2 (\sec x \tan x) e^u \\ &= 2x e^{\sec x} + x^2 \sec x \tan x e^{\sec x} \end{aligned}$$

(b) (4 points) For  $f(t) = \cos\left(\frac{e^t - 2}{\arcsin t}\right)$ , find  $f'(t)$ .

This is a chain rule problem whose internal calculation is itself a quotient rule. Let us pre-emptively assign  $u = \frac{e^t - 2}{\arcsin t}$ , and then

$$\begin{aligned} f'(t) &= \frac{d}{dt} \cos\left(\frac{e^t - 2}{\arcsin t}\right) \\ &= \frac{d}{dt} \cos u \\ &= \frac{du}{dt} \frac{d}{du} \cos u \\ &= \left( \frac{d}{dt} \frac{e^t - 2}{\arcsin t} \right) \frac{d}{du} \cos u \\ &= \left( \frac{\arcsin t \frac{d}{dt}(e^t - 2) - (e^t - 2) \frac{d}{dt} \arcsin t}{(\arcsin t)^2} \right) (-\sin u) \\ &= \left( \frac{(\arcsin t)e^t - (e^t - 2) \frac{1}{\sqrt{1-t^2}}}{(\arcsin t)^2} \right) \left( -\sin \frac{e^t - 2}{\arcsin t} \right) \end{aligned}$$

(c) (4 points) Compute  $\frac{d}{d\theta} \sin(\theta + \ln \theta)$ .

This is a chain rule problem; let  $u = \theta + \ln \theta$ , and then:

$$\begin{aligned} \frac{d}{d\theta} \sin(\theta + \ln \theta) &= \frac{d}{d\theta} \sin u \\ &= \frac{du}{d\theta} \frac{d}{du} \sin u \\ &= \left( 1 + \frac{1}{\theta} \right) \cos u \\ &= \left( 1 + \frac{1}{\theta} \right) \cos(\theta + \ln \theta) \end{aligned}$$

2. (4 points) Given the equality  $x \sin y = 1 - y^2$ , find a formula for  $\frac{dy}{dx}$  by implicit differentiation.

Differentiating both sides of the equation with respect to  $x$ :

$$\begin{aligned}\frac{d}{dx}(x \sin y) &= \frac{d}{dx}(1 - y^2) \\ \left(\frac{d}{dx}x\right) \sin y + x \frac{d}{dx} \sin y &= 0 - \frac{d}{dx}y^2 \\ \sin y + x \frac{dy}{dx} \frac{d}{dy} \sin y &= -\frac{dy}{dx} \frac{d}{dy} y^2 \\ \sin y + x \frac{dy}{dx} \cos y &= -\frac{dy}{dx} 2y \\ x \frac{dy}{dx} \cos y + 2y \frac{dy}{dx} &= -\sin y \\ (x \cos y + 2y) \frac{dy}{dx} &= -\sin y \\ \frac{dy}{dx} &= \frac{-\sin y}{x \cos y + 2y}\end{aligned}$$

3. (4 points) An air compressor which delivers 10 cubic feet per minute is being used to inflate a spherical balloon of volume  $V$  and radius  $r$  which is currently 8 feet in radius; we wish to know how quickly the radius is increasing. Calculate  $\frac{dr}{dt}$  below, using the facts:  $V = \frac{4}{3}\pi r^3$ ,  $\frac{dV}{dt} = 10$ ,  $r = 8$  currently.

We differentiate both sides of the expression  $V = \frac{4}{3}\pi r^3$  with respect to  $t$ , and then algebraically isolate  $\frac{dr}{dt}$ .

$$\begin{aligned}\frac{d}{dt}V &= \frac{d}{dt} \frac{4}{3}\pi r^3 \\ \frac{dV}{dt} &= \frac{dr}{dt} \frac{d}{dr} \left(\frac{4}{3}\pi r^3\right) \\ \frac{dV}{dt} &= \frac{dr}{dt} 4\pi r^2 \\ \frac{dV}{dt} &= \frac{dr}{dt} 4\pi r^2\end{aligned}$$

and since  $\frac{dV}{dt} = 10$  and the current value of  $r$  is 8, we may conclude that the current value of  $\frac{dr}{dt}$  is  $\frac{10}{4\pi(8^2)} = \frac{10}{256\pi} = \frac{5}{128\pi}$ .

4. (2 point bonus) If  $n$  is a positive integer, find a general formula for  $\frac{d^n}{dx^n}(x^{n-1} \ln x)$  on the back of this page.

We might look at some specific examples, deriving each example from its predecessor:

$$\begin{aligned}\frac{d}{dx} \ln x &= \frac{1}{x} \\ \frac{d^2}{dx^2} (x \ln x) &= \frac{d}{dx} \left( \ln x + \frac{x}{x} \right) = \frac{d}{dx} (\ln x + 1) \\ &= \frac{1}{x} \\ \frac{d^3}{dx^3} (x^2 \ln x) &= \frac{d^2}{dx^2} \left( 2x \ln x + \frac{x^2}{x} \right) = \frac{d^2}{dx^2} (2x \ln x + x) \\ &= 2 \cdot \frac{1}{x} = \frac{2}{x} \\ \frac{d^4}{dx^4} (x^3 \ln x) &= \frac{d^3}{dx^3} \left( 3x^2 \ln x + \frac{x^3}{x} \right) = \frac{d^3}{dx^3} (3x^2 \ln x + x^2) \\ &= 3 \cdot \frac{2}{x} = \frac{6}{x} \\ \frac{d^5}{dx^5} (x^4 \ln x) &= \frac{d^4}{dx^4} \left( 4x^3 \ln x + \frac{x^4}{x} \right) = \frac{d^4}{dx^4} (4x^3 \ln x + x^3) \\ &= 4 \cdot \frac{6}{x} = \frac{24}{x}\end{aligned}$$

and the pattern that emerges is that after taking one of the  $n$  derivatives requested of  $x^{n-1} \ln x$ , we are left with  $\frac{d^{n-1}}{dx^{n-1}} ((n-1)x^{n-2} \ln x + x^{n-2})$ . The second term of this sum can be easily seen to be zero — hit  $x^{n-2}$  with  $n-1$  derivatives and it will reach zero — while the first term of this sum looks a lot like the original value we were looking at, but with all exponents decreased by 1. If we repeat this process  $n-1$  times, we would get that

$$\frac{d^n}{dx^n} (x^{n-1} \ln x) = \frac{(n-1)(n-2)(n-3) \cdots 2 \cdot 1}{x}$$