This test is closed-book and closed-notes. No calculator is allowed for this test. For full credit show all of your work (legibly!), unless otherwise specified.

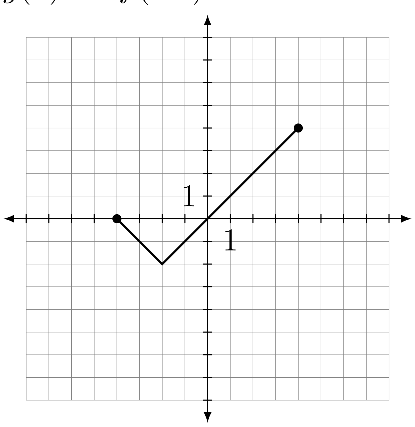
The problems are in no particular order, and it is suggested that you look at all of them before beginning to answer any.

1. **(10 points)** *Answer the following questions about graphs.*

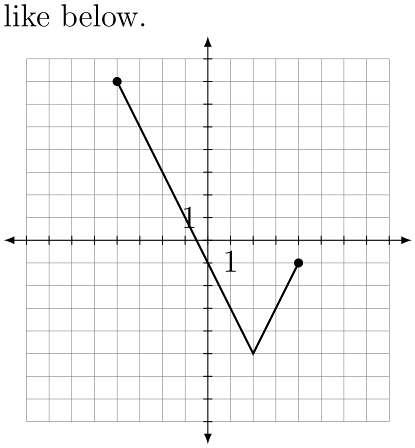
   (a) **(3 points)** *Given the piecewise function* $g(x) = \begin{cases} x^2 - 2 \text{ if } & x > 1 \\ 2x - 5 \text{ if } & x \leq 1 \end{cases}$, *calculate the following values:*

   - $g(3)$.
     Since $3 > 1$, $g(3) = 3^2 - 2 = 7$.
   - $g(2)$.
     Since $2 > 1$, $g(2) = 2^2 - 2 = 2$.
   - $g(1)$.
     Since $1 \leq 1$, $g(1) = 2 \cdot 1 - 5 = -3$.

   (b) **(4 points)** *For* $f(x)$ *as shown on the graph, sketch the graph of its transformation* $g(x) = 2f(-x) - 1$.



The transformation includes, in order, a horizontal flip (due to the appearance of $f(-x)$), a vertical stretch by a factor of 2 (since the function result is multiplied by 2) and a shift downwards by 1 (since 1 is subtracted from the function value. We see what this looks like below.



   (c) **(3 points)** *Determine the equation of the line through the points* $(1, 4)$ *and* $(4, -5)$.

We start by calculating the slope of the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-4}{4-1} = \frac{-9}{3} = -3$$

and then, using one of the points — in this example $(1, 4)$, although the choice of point doesn't matter — to get the equation of the line in point-slope form:

$$(y-4) = -3(x-1)$$

which can, if desired, be algebraically rearranged to be in slope-intercept form:

$$y = -3x + 7$$

2. **(10 points)** *The Hong Kong Cavaliers have 1000 fans who would come to a concert if tickets cost \$10. Polling of the fanbase suggests that every increase in the ticket price by \$1 would reduce attendance by 50.*

   (a) **(4 points)** *Express the number of concertgoers as a function of the ticket price.*

   Let us denote the ticket price by x; we have a base attendance of 1000, and we diminish that by 50 for eachdollar by which x exceeds 10; in otherwise, we must take $50(x-10)$ people away from that base attendance to determine the attendance at a ticket price x. Thus our demand function is $D(x) = 1000 - 50(x-10)$. This may, but need not, be algebraically simplified to $D(x) = 1500 - 50x$.

   (b) **(2 points)** *Express the total revenue from ticket sales as a function of the ticket price.*

   As seen above, we sell $1500-50x$ tickets at ticket price x; our total revenue from so doing would be the product of the demand and the price, or $R(x) = D(x) \cdot x = (1500 - 50x)x$. This may, but need not, be expanded algebraically to give $R(x) = -50x^2 + 1500x$.

   (c) **(4 points)** *Find the ticket price which maximizes revenue.*

   We seek to find the value of x maximizing the quadratic $R(x) = -50x^2 + 1500x$ determined in the previous section; to do so we will locate its vertex, which is most easily done by placing $R(x)$ in standard form:

$$\begin{aligned} R(x) &= -50x^2 + 1500x \\ &= -50(x^2 - 30x) \\ &= -50(x^2 - 30x + 15^2 - 15^2) \\ &= -50(x^2 - 30x + 15^2) + 50 \cdot 15^2 \\ &= -50(x-15)^2 + 50 \cdot 15^2 \end{aligned}$$

   so while k here is a rather unpleasant number $(50 \cdot 15^2 = 11250)$, we never actually need to calculate it; we're interested in the x-coordinate of the vertex, which indicates that the minimizing ticket price is \$15.

3. **(15 points)** *Answer the following questions about the functions* $f(x) = \frac{2}{x+1}$ *and* $g(x) = \frac{x}{x+2}$. *In each question asking for multiple answers,* label which is which.

(a) **(3 points)** *Find the inverse of the function* $f(x)$.

We know that $f(f^{-1}(x)) = x$, and by expanding $f(x)$ we can express $f^{-1}(x)$ in terms of x:

$$\begin{aligned} x &= f(f^{-1}(x)) \\ x &= \frac{2}{f^{-1}(x)+1} \\ (f^{-1}(x)+1)x &= 2 \\ f^{-1}(x)+1 &= \frac{2}{x} \\ f^{-1}(x) &= \frac{2}{x} - 1 \end{aligned}$$

(b) **(2 points)** *Write formulas, which need not be simplified, for* $(g-f)(x)$ *and* $\frac{f}{g}(x)$.

$$\begin{aligned} (g-f)(x) = g(x) - f(x) &= \frac{x}{x+2} - \frac{2}{x+1} \\ \frac{f}{g}(x) = \frac{f(x)}{g(x)} &= \frac{\frac{2}{x+1}}{\frac{x}{x+2}} \end{aligned}$$

(c) **(3 points)** *Write formulas, which need not be simplified, for* $f(g(x))$ *and* $f(f(x))$.

$$\begin{aligned} f(g(x)) &= f\left(\frac{x}{x+2}\right) = \frac{2}{\frac{x}{x+2}+1} \\ f(f(x)) &= f\left(\frac{2}{x+1}\right) = \frac{2}{\frac{2}{x+1}+1} \end{aligned}$$

(d) **(3 points)** *Determine the domains of* $f(x)$ *and* $g(x)$.

Since $f(x)$ has a denominator of $x+2$ (but is otherwise free of "dangerous" calculations), we know that the domain of $f(x)$ consists of those points where $x+2 \neq 0$, or where $x \neq -2$; if desired, this can be expressed as the interval form $(-\infty,-2) \cup (-2,\infty)$.

Likewise, $g(x)$ has a denominator of $x+1$, but is otherwise free of dangerous calculations, so we know that the domain of $g(x)$ consists of those points where $x+1 \neq 0$, or where $x \neq -1$; if desired, this can be expressed as the interval form $(-\infty,-1) \cup (-1,\infty)$.

(e) **(4 points)** *Determine the domains of* $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, *and* $\frac{f}{g}(x)$.

We recall that $f+g$, $f-g$, and fg have domains given by the overlap of the domains of f and g, so all three of these functions have domains given by $x \neq -1$ and $x \neq -2$; if desired, this can be expressed as the interval form $(-\infty,-2) \cup (-2,-1) \cup (-1,\infty)$.

However, $\frac{f}{g}$ has an additional restriction: in addition to needing x-values in the domain of f and g, $\frac{f}{g}$ is only defined where $g(x) \neq 0$. Since $g(x) = \frac{x}{x+2}$, it is clear that $g(x) = 0$ when $x = 0$, so we must exclude 0 from the domain of $\frac{f}{g}$; thus $\frac{f}{g}$ has a domain given by the conditions $x \neq 0$, $x \neq -1$, and $x \neq -2$, or in interal form, $(-\infty,-2) \cup (-2,-1) \cup (-1,0) \cup (0,\infty)$.

4. **(15 points)** *Perform the following arithmetic and algebraic operations.*

   (a) **(3 points)** *Factor the quadratic* $x^2 - 8x + 12$.
   Trial and error suggests the possibile factorizations $(x-1)(x-12)$, $(x-2)(x-6)$, $(x-3)(x-4)$, $(x+1)(x+12)$, $(x+2)(x+6)$, and $(x+3)(x+4)$. The second of these is correct. This can also be found using the quadratic formula to note that $x^2 - 8x + 12$ is zero when $x = 2$ and $x = 6$.

   (b) **(3 points)** *Expand and simplify the polynomial* $(x^3 + 5) - (2x - 1)(x^2 + 3x)$.
   Using straightforward algebraic techniques:

$$\begin{aligned}(x^3+5)-(2x-1)(x^2+3x) &= (x^3+5)-(2x^3+6x^2-x^2-3x)\\ &= -x^3-5x^2+3x+5\end{aligned}$$

   (c) **(3 points)** *Calculate* $16^{3/4}$. Exploding the rational exponent into two individually calculable steps, we get:

$$16^{3/4} = \left(16^{1/4}\right)^3 = \sqrt[4]{16}^3 = 2^3 = 8$$

   (d) **(3 points)** *Simplify the expression* $\frac{x}{x-4} - \frac{3}{x+6}$.
   We find a common denominator and simplify the numerator:

$$\begin{aligned}\frac{x}{x-4}-\frac{3}{x+6} &= \frac{x(x+6)}{(x-4)(x+6)}-\frac{(x-4)3}{(x-4)(x+6)}\\ &= \frac{x(x+6)-(x-4)3}{(x-4)(x+6)}\\ &= \frac{x^2+6x-(3x-12)}{x^2+2x-24}\\ &= \frac{x^2+3x+12}{x^2+2x-24}\end{aligned}$$

   The denominator could be left factored; expansion versus factorization is a stylistic variation within the bounds of what is considered "simplified".

   (e) **(3 points)** *Simplify the expression* $\frac{(2x^3)^2(3x^4)}{(x^3)^4}$.
   Using exponential distribution-and-gathering techniques:

$$\frac{(2x^3)^2(3x^4)}{(x^3)^4} = \frac{2^2x^6\cdot 3x^4}{x^{12}} = \frac{12x^{10}}{x^{12}} = \frac{12}{x^2}$$

   One might alternatively express the final answer as $12x^{-2}$.

5. **(10 points)** *Answer the following questions about the quadratic* $q(x) = 6x^2 + 12x - 5$.

   (a) **(2 points)** *What is the average rate of change of the function* $q(x)$ *between the points* $x = -1$ *and* $x = 1$*?*
   The average value is given by the quotient

$$\frac{q(1)-q(-1)}{1-(-1)} = \frac{(6\cdot 1^2+12\cdot 1-5)-(6(-1)^2+12(-1)-5)}{2} = \frac{13-(-11)}{2} = \frac{24}{2} = 12$$

(b) **(3 points)** *Put the quadratic $q(x)$ in standard form.*

$$\begin{aligned} q(x) &= 6x^2 + 12x - 5 \\ &= 6(x^2 + 2x) - 5 \\ &= 6(x^2 + 2x + 1 - 1) - 5 \\ &= 6(x^2 + 2x + 1) - 6 - 5 \\ &= 6(x+1)^2 - 11 \end{aligned}$$

(c) **(1 point)** *Does $q(x)$ have a maximum or minimum value? If so, identify which it is and what its value is.*

Its vertex is at $(-1, -11)$; since this is a quadratic with positive a, it curves upwards and this vertex is the lwoest point on the graph; thus -11 is the minimum value achieved by $q(x)$.

(d) **(4 points)** *Determine the vertex of this quadratic function, its x-intercepts if they exist, and its y-intercept. Label which is which.*

The vertex, as mentioned above, is at $(-1, -11)$, as demonstrated by the standard form representation. The y-intercept is simply $q(0) = 6 \cdot 0^2 + 12 \cdot 0 - 5 = -5$. The x-intercepts are given by finding the solutions tot he equation $q(x) = 0$; since this equation is $6x^2 + 12x - 5 = 0$, it is easily solved using the quadratic equation:

$$x = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 6(-5)}}{2 \cdot 6} = \frac{-12 \pm \sqrt{264}}{12}$$

This can, but need not, be simplified to $1 \pm \frac{\sqrt{66}}{6}$.