For full credit show all of your work (legibly!), unless otherwise specified. Answers should be simplified down to arithmetic expressions whenever possible — only *unsimplifiable* trigonometric and exponential functions may be left unevaluated.

1. **(10 points)** *Answer the following questions about series.*

   (a) **(2 points)** *Express* $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{20}$ *in sigma notation.*
   We are taking expressions of the form $\frac{1}{n}$ and adding them up as n ranges from 2 to 20, so the sigma-notation representation of this sum is $\sum_{n=2}^{20}\frac{1}{n}$. Another letter may be used in place of n (n, i, j, and k are traditional, but any letter is valid).

   (b) **(4 points)** *Calculate the arithmetic series partial sum* $3+7+11+15+19+\cdots+79$.
   First, we must calculate how many terms this partial sum has. The first term is 3 and the common difference is 4, so the terms are given by the formula $a_n = 3+4(n-1)$. To determine the index on the last term:

   $$\begin{aligned}79 &= 3+4(n-1)\\ 76 &= 4(n-1)\\ 19 &= n-1\\ 20 &= n\end{aligned}$$

   so there are 20 terms in total. We can now find the sum either by the series-reversal trick:

   $$\begin{aligned}S &= 3+7+\cdots+79\\ S &= 79+75+\cdots+3\\ 2S &= 82+82+\cdots+82 = 20\cdot 82\end{aligned}$$

   so $S=\frac{20\cdot 82}{2}=820$, or by the sum formula:

   $$S = a_1 n + \frac{n(n-1)}{2}d = 3\cdot 20 + \frac{20\cdot 19}{2}4 = 60+760 = 820$$

   (c) **(2 points)** *Evaluate* $\sum_{k=4}^{7}(k^2-k)$.
   This sum consists of only four terms, and is easily expanded:

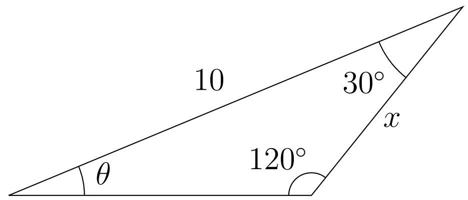
   $$\sum_{k=4}^{7}(k^2-k) = (4^2-4)+(5^2-5)+(6^2-6)+(7^2-7) = 12+20+30+42 = 104$$

   (d) **(2 points)** *Evaluate the geometric series infinite sum* $4-1+\frac{1}{4}-\frac{1}{16}+\frac{1}{64}-\frac{1}{256}+\cdots$
   This series is geometric with first term 4 and common ratio $-\frac{1}{4}$, so we use the formula:

   $$S = \frac{a_1}{1-r} = \frac{4}{1-\left(-\frac{1}{4}\right)} = \frac{4}{\frac{5}{4}} = \frac{16}{5}$$

2. **(10 points)** *Calculate the labeled quantities in the triangles (not drawn to scale) below.*

   (a) **(5 points)** *Determine x and θ:*



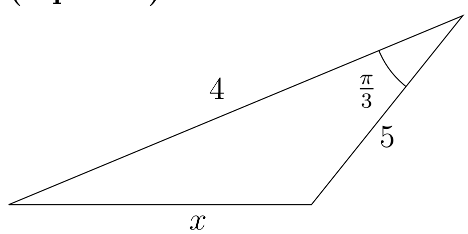
Since the angles add up to 180°, we know that $120 + 30 + \theta = 180$, so $\theta = 30°$.

In this triangle two angles and only one side have been specified, making it a prime candidate for application of the Rule of Sines. Looking at the two labeled sides and their opposite angles, we get:

$$\frac{x}{\sin\theta} = \frac{10}{\sin 120°}$$

which, multiplying both sides by $\sin\theta = \sin 30°$, gives $x = \frac{10 \sin 30°}{\sin 120°} = \frac{10\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{10}{\sqrt{3}}$ or $\frac{10\sqrt{3}}{3}$.

   (b) **(5 points)** *Determine x:*



Since we have only one labeled angle but know two of the sides, the relationship among the sides is best discovered by the Rule of Cosines; we use the known angle and the two known adjacent sides to determine that:

$$\begin{aligned} x^2 &= 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cdot \cos\frac{\pi}{3} \\ &= 16 + 25 - 40 \cdot \frac{1}{2} = 21 \\ x &= \sqrt{21} \end{aligned}$$

Note that the negative square root can be ignored, since r is a length, and as such is presumably positive.

3. **(12 points)** *Answer the following questions about trigonometric equations.*

   (a) **(4 points)** *Find* all *solutions to the equation* $6\sin(2x) = -3\sqrt{3}$.

Note below the procedure used to invert the sine, to ensure that *all* solutions are found:

$$\begin{aligned}
6\sin(2x) &= -3\sqrt{3} \\
\sin(2x) &= \frac{-\sqrt{3}}{2} \\
2x &= \arcsin\frac{-\sqrt{3}}{2} + 2\pi n \text{ or } \left(\pi - \arcsin\frac{-\sqrt{3}}{2}\right) + 2\pi n \\
2x &= \frac{-\pi}{3} + 2\pi n \text{ or } \frac{4\pi}{3} + 2\pi n \\
x &= \frac{-\pi}{6} + \pi n \text{ or } \frac{2\pi}{3} + \pi n
\end{aligned}$$

So x can take on any value of the form $\pi n - \frac{\pi}{6}$ or of the form $\pi n + \frac{2\pi}{3}$.

(b) **(4 points)** *Find* any one *solution to the equation* $4\sec(3x) - 2 = 6$.

Since we were only asked to find one solution, we don't have to be too cautious with our cosine-inversion, and can use the ordinary arc-cosine:

$$\begin{aligned}
4\sec(3x) - 2 &= 6 \\
\frac{4}{\cos(3x)} - 2 &= 6 \\
\frac{4}{\cos(3x)} &= 8 \\
4 &= 8\cos(3x) \\
\frac{1}{2} &= \cos(3x) \\
\arccos\frac{1}{2} &= 3x \\
\frac{\pi}{3} &= 3x \\
\frac{\pi}{9} &= x
\end{aligned}$$

There are several other acceptable answers, if a different choice of value whose cosine is $\frac{1}{2}$ is used.

(c) **(4 points)** *Verify the trigonometric identity* $\frac{1-\sin x}{1+\sin x} = (\sec x - \tan x)^2$.

We expand the right side:

$$\begin{aligned}(\sec x - \tan x)^2 &= \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)^2 \\ &= \left(\frac{1-\sin x}{\cos x}\right)^2 \\ &= \frac{(1-\sin x)^2}{\cos^2 x} \\ &= \frac{(1-\sin x)^2}{1-\sin^2 x} \\ &= \frac{(1-\sin x)^2}{(1-\sin x)(1+\sin x)} = \frac{1-\sin x}{1+\sin x}\end{aligned}$$

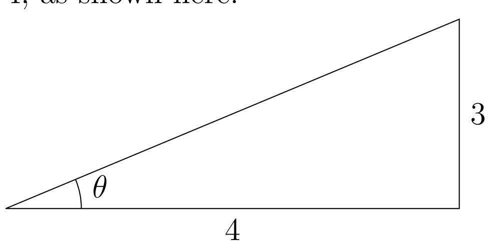
4. **(10 points)** *Answer the following questions about evaluating trigonometric expressions.*

(a) **(2 points)** *Evaluate* $\arctan(-1)$.

There are a few familiar tangents, and their associated arctangents should be discernable from the familiarity of their form. For instance, we know that $\tan\frac{-\pi}{4} = -1$, and thus $\arctan(-1) = \frac{-\pi}{4}$.

(b) **(4 points)** *Evaluate* $\csc\big(\arctan(\frac{3}{4})\big)$.

We give a simple name to $\arctan\frac{3}{4}$; traditionally we might call it θ. To convey this information we might build a triangle exemplifying this relationship between θ and $\frac{3}{4}$, which we might write as $\theta = \arctan\frac{3}{4}$, but more comprehensibly as $\tan\theta = \frac{3}{4}$; in a right triangle with θ as one of the angles, we know that $\tan\theta$ represents the ratio of the lengths of the opposite side and the adjacent side. We would thus represent this relationship by making the opposite side of the triangle have length 3, and the adjacent side have length 4, as shown here:



Furthermore, the hypotenuse, which is not labeled in the above picture, can be calculated by the Pythagorean Theorem to have length $\sqrt{3^2+4^2} = \sqrt{9-16} = \sqrt{25} = 5$.

After that setup, our actual calculation ends up being very easy! We wanted to find $\csc\arctan\frac{3}{4}$, which in light of our definition of θ can be written as simply $\csc\theta$. Finding any trigonometric identity of an angle in a labeled right triangle is simply a matter of dividing the appropriate sides: the cosecant is the ratio of the lengths of the hypotenuse and opposite side, so in this case, $\csc\theta = \frac{5}{3}$.

(c) **(2 points)** *Evaluate* $\arcsin\frac{1}{2}$.

There are a few familiar sines, and their associated arcsines should be discernable from the familiarity of their form. Recall that the arcsine function returns values between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$. We know that $\cos\frac{\pi}{6} = \frac{1}{2}$, and thus $\arcsin\frac{1}{2} = \frac{-\pi}{2}$.

(d) **(3 points)** *Evaluate the expression* $\cos(55°)\cos(10°) + \sin(55°)\sin(10°)$.
The above expression is a template which is recognizable as one of those covered in our angle-addition rules; recall that

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

whose right side matches nicely with the expression seen here, so

$$\cos(55°)\cos(10°) + \sin(55°)\sin(10°) = \cos(55° - 10°) = \cos 45° = \frac{\sqrt{2}}{2}$$

5. **(8 points)** *Identify each of the following sequences as arithmetic, geometric, or neither, and find a formula for each sequence.*

(a) **(2 points)** $1, 4, 9, 16, 25, \ldots$.
Note that the differences of the 4 pairs of consecutive terms seen here are 3, 5, 7, and 9; since these are the not the same, this sequence is not arithmetic. The ratios of the 4 pairs of consecutive terms seen here are 4, $\frac{9}{4}$, $\frac{16}{9}$, and $\frac{25}{16}$; since these are not the same, this sequence is not geometric. This is thus a sequence which is neither arithmetic nor geometric, but inspection of the pattern will reveal it to be given by the formula $a_n = n^2$.

(b) **(2 points)** $27, 9, 3, 1, \frac{1}{3}, \ldots$.
Note that the differences of the 4 pairs of consecutive terms seen here are 18, 6, 2, and $\frac{2}{3}$; these are not the same, so this sequence is not arithmetic. The ratios of the 4 pairs of consecutive terms seen here, however, are $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, and $\frac{1}{3}$; since these are the same, this sequence is geometric with common ratio $\frac{1}{3}$ and first term 27, and thus has formula $a_n = 27\left(\frac{1}{3}\right)^{n-1}$.

(c) **(2 points)** $5, 2, -1, -4, -7, -10, \ldots$.
Note that the differences of the 5 pairs of consecutive terms seen here are −3, −3, −3, −3, and −3; since these are the same, this sequence is arithmetic with common difference −3 and first term 5, and thus has formula $a_n = 5 - 3(n - 1)$.

(d) **(2 points)** $3, -3, 3, -3, 3, -3, \ldots$.
Note that the differences of the 5 pairs of consecutive terms seen here are −6, 6, −6, 6, and −6; these are not the same, so this sequence is not arithmetic. The ratios of the 5 pairs of consecutive terms seen here, however, are −1, −1, −1, −1, and −1; since these are the same, this sequence is geometric with common ratio −1 and first term 3, and thus has formula $a_n = 3(-1)^{n-1}$.

6. **(10 points)** *Answer the following questions about sequence exploration.*

(a) **(4 points)** *The third term of an arithmetic sequence is* 19 *and the seventh term is* 3. *What is the formula for the sequence?*
We set up the template for an arithmetic sequence: $a_n = a_1 + d(n - 1)$. Then $19 = a_3 = a_1 + 2d$ and $3 = a_7 = a_1 + 6d$. Thus:

$$\begin{aligned} 19 &= a_1 + 2d \\ 3 &= a_1 + 6d \\ (19 - 3) &= (a_1 + 2d) - (a_1 + 6d) \\ 16 &= -4d \\ -4 &= d \end{aligned}$$

and now we can find $19 = a_1 + 2(-4)$ so $a_1 = 19 + 8 = 27$. Thus, $a_n = 27 - 4(n-1)$.

(b) **(3 points)** *An arithmetic sequence has formula* $a_n = 6 + 9(n-1)$. *Which term in this sequence is equal to* 114*?*

We let $114 = 6+9(n-1)$ and solve for n: $108 = 9(n-1)$, so $n-1 = 12$ and then $n = 13$; thus 114 is the thirteenth term of the sequence.

(c) **(3 points)** *The first term of a geometric sequence is* 9*, and the second term is* -3. *What is the sixth term?*

The first term of this geometric sequence is 9; the common ratio is $\frac{9}{-3} = \frac{-1}{3}$, so the formula is $a_n = 9\left(\frac{-1}{3}\right)^{n-1}$. Then $a_6 = 9\left(\frac{-1}{3}\right)^5 = 9 \cdot \frac{-1}{243} = \frac{-1}{27}$.