1. **(10 points)** *Answer the following questions about growth and decay.*

   (a) **(3 points)** *Market analysis suggests than an investment in Tessier-Ashpool S.A. will have a relative growth rate of 6% per year. If you invest \$5000 in this corporation, how long will it take your investment to grow to a value of \$6000?*

   The model for the growth of a \$5000 investment with a 6% rate of growth is $f(t) = 5000e^{1.06t}$, and here we are asking which value of t yields $f(t) = 6000$. Thus:

$$\begin{aligned} 5000e^{0.06t} &= 6000 \\ e^{0.06t} &= \frac{6000}{5000} = \frac{6}{5} \\ 0.06t &= \ln\frac{6}{5} \\ t = \frac{\ln\frac{6}{5}}{0.06} &\approx 3.04 \text{ years} \end{aligned}$$

   The approximation step is provided for context, and need not be calculated under exam conditions.

   (b) **(3 points)** *Miraclo is a drug with a half-life in the body of 5 hours. Produce a function describing the quantity still in the body t hours after administration of a 40mg dose.*

   A decay function with a given half-life τ is given by either $f(t) = A\left(\frac{1}{2}\right)^{\frac{t}{\tau}}$ or $f(t) = Ae^{\frac{-\ln 2}{\tau}t}$, so in this case we would produce the function $f(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{5}}$ or $f(t) = 40e^{\frac{-\ln 2}{5}t}$. Note that these two functions are actually algebraically equivalent.

   Alternatively, if one dislikes appeal to the half-life formula, one can start with an arbitrary exponential function template $f(t) = Ae^{rt}$ and use the fact $f(0) = 40$ to find $A = 40$, and the fact $f(5) = 20$ to find:

$$\begin{aligned} 20 = f(5) &= 40e^{r\cdot 5} \\ \frac{1}{2} &= e^{5r} \\ \ln\frac{1}{2} &= 5r \\ \frac{1}{5}\ln\frac{1}{2} &= r \end{aligned}$$

   (c) **(2 points)** *Miraclo becomes ineffective when there is less than 15mg in the body. Using your result from the above question, determine how long after taking a 40mg dose this will occur.*

   From the above formula in exponential form, solving for $f(t) = 15$, we get

$$\begin{aligned} 40e^{\frac{-\ln 2}{5}t} &= 15 \\ e^{\frac{-\ln 2}{5}t} &= \frac{15}{40} = \frac{3}{8} \\ \frac{-\ln 2}{5}t &= \ln\frac{3}{8} \\ t &= \frac{-5\ln\frac{3}{8}}{\ln 2} \approx 7.07 \text{ hours} \end{aligned}$$

The approximation step is provided for context, and need not be calculated under exam conditions.

(d) **(2 points)** *An alien spacecraft, red-hot from its entry into the atmosphere, lands on a warm summer day. Its temperature in degrees Fahrenheit t minutes after impact is given by the function $f(t) = 80 + 1600e^{-0.015t}$. A science team can begin experiments on it after it has cooled to $400°F$. How many minutes will they need to wait?*

We are solving for the value of t such that $f(t) = 400$:

$$\begin{aligned} 80 + 1600e^{-0.015t} &= 400 \\ 1600e^{-0.015t} &= 320 \\ e^{-0.015t} &= \frac{320}{1600} = \frac{1}{5} \\ -0.015t &= \ln\frac{1}{5} \\ t &= \frac{\ln\frac{1}{5}}{-0.015} \approx 107.3 \text{ minutes} \end{aligned}$$

The approximation step is provided for context, and need not be calculated under exam conditions.

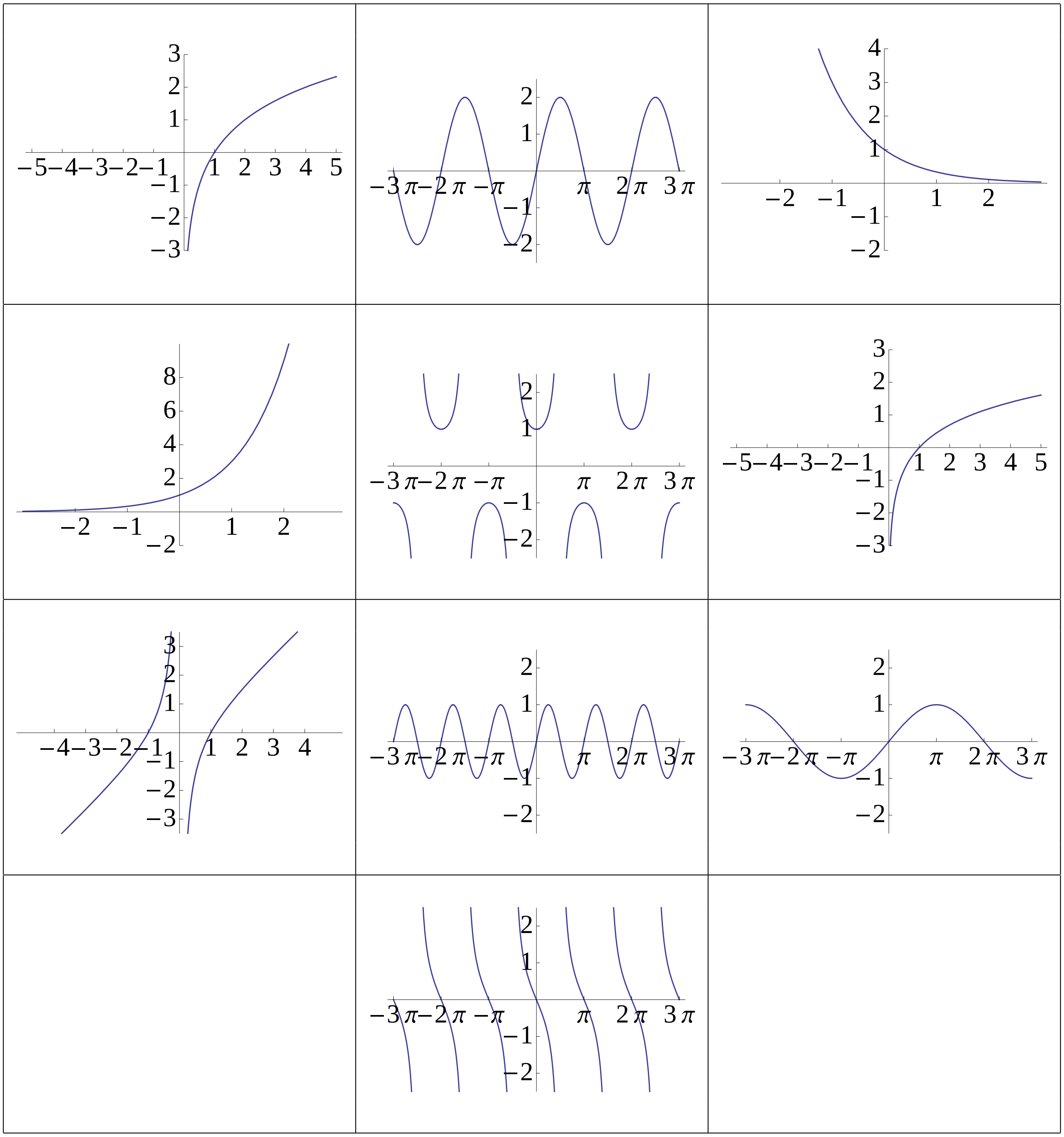
2. **(10 points)** *The following ten graphs are of the following functions:*

$$A(x) = 3^x \qquad B(x) = \left(\frac{2}{3}\right)^x \qquad C(x) = \log_2 x \qquad D(x) = \sin 2x$$

$$E(x) = 2\sin x \qquad F(x) = \sin\frac{x}{2} \qquad G(x) = \tan(-x) \qquad H(x) = \sec x$$

$$I(x) = \frac{x}{x^2 - 1} \qquad J(x) = \frac{x^2 - 1}{x}$$

Label each picture with the letter of the appropriate function.



Several of the graphs have highly characteristic and unique shapes which we can identify merely on sight. Graph #1 has an asymptote at 0 and exists only in the positive half of the plane; this is characteristic of a logarithm so it must be $C(x)$. Graph #5 has the disjoint humps and central gap of a secant-like function, so it must be $H(x)$. Graph #10 has the snaky repetition of a tangent-like function, and thus must be $G(x)$.

Of the remainder, we can broadly classify then into sinusoidal, exponential, and "other". Graphs #2, #8, and #9 are sinusoidal and must thus correspond in some order to $D(x)$, $E(x)$, and $F(x)$. Graph #2 has amplitude of 2, which corresponds to a multiplication by 2 outside of the trig function, as appears in $E(x)$. Graph #8 has period of π and graph 9 has period of 4π, so they are respectively the result of a horizontal compression and a stretch of the ordinary sine function: we see that $D(x)$ is a compression and $F(x)$ is a stretch.

Now we handle the exponentials, of which there are two: graph #4 resembles a growth curve while graph #6 is a decay curve. On the functional side, there is $A(x)$, which has a base larger than 1 and thus is a growth curve, while $B(x)$ has a base less than 1 and represents decay.

Finally, the two oddities are graphs #3 and #7, which we recognize as corresponding in some order to the rational functions $I(x)$ and $J(x)$. The easiest visual identification is by vertical asymptotes; $I(x)$ has $x^2 - 1$ in the denominator and must have asymptotes at $x = -1$ and $x = 1$, as graph #3 does, while $J(x)$ has x in the denominator and has an asymptote at $x = 0$, like graph #7.

3. **(10 points)** *Answer the following questions about trigonometry.*

   (a) **(3 points)** *Evaluate* $\csc \frac{17\pi}{6}$.
   Note that $\frac{17\pi}{6} = (2\frac{5}{6})\pi$, so $\csc \frac{17\pi}{6} = \csc \frac{5\pi}{6} = \frac{1}{\sin \frac{5\pi}{6}}$. The angle $frac5\pi6$ describes a point in the second quadrant, in which the sine in positive, so $\sin \frac{5\pi}{6}$ is positive, and since this argument is a fully reduced fraction of the form $\frac{k\pi}{6}$, we know that $\sin \frac{5\pi}{6} = \pm\frac{1}{2}$; since it is positive, we know $\csc \frac{17\pi}{6} = \frac{1}{1/2} = 2$.

   (b) **(4 points)** *Identify the period, amplitude, and vertical shift of the periodic function* $g(x) = \frac{1}{3}\cos(\frac{3}{5}x) + 8$.
   The vertical shift can be clearly seen to be +8, since the addition of 8 to the function value enacts a vertical shift of 8 on the graph.
   The amplitude and period are dictated by the vertical and horizontal stretch factors respectively: we see that this graph is vertically stretched by a factor of $\frac{1}{3}$ (i.e. compressed by a factor of 3), so the amplitude of the standard cosine function, which is 1, is transformed to $\frac{1}{3}$.
   The period is dictated by the compression factor indicated in the argument to the cosine. We compress the ordinary cosine curve horizontally by a factor of $\frac{3}{5}$, so the standard period of 2π is transformed into a period of length $\frac{2\pi}{3/5} = \frac{10\pi}{3}$.

   (c) **(3 points)** *If* $\csc x = \frac{-13}{12}$ *and x describes a terminal point in the second quadrant, what is* $\cot x$*?*
   One of our trigonometric identities is $\sin^2 x+\cos^2 x = 1$; dividing both sides by $\sin^2 x$ gives $1 + \cot^2 x = \csc^2 x$. Thus, $\cot^2 x = \csc^2 x - 1 = \frac{169}{144} - 1 = \frac{25}{144}$, so $\cot x = \pm\sqrt{\frac{25}{144}} = \pm\frac{5}{12}$. Since x is in the second quadrant, $\cot x$ is negative, so $\cot x = -\frac{5}{12}$.

4. **(10 points)** *Answer the following questions about polynomial functions.*

   (a) **(5 points)** *Identify the x-intercepts, y-intecept, and long-term behavior of* $f(x) = 2(x - 3)(x + 4)(x + 1)$.
   The y-intercept is simply $f(0) = 2(-3)(4)(1) = -24$.
   Now we identify zeroes. The factorization above makes it clear that $f(-4) = f(-1) = f(3) = 0$, so the zeroes are -4, -1, and 3.
   The leading term of this on expansion would be $2x^3$, so we know the long-term behavior of this function has an upwards-facing arrow on the right side and a downwards-facing arrow on the left side, as befits a polynomial whose leading term has an odd exponent and a positive coefficient.

(b) **(3 points)** *Using either synthetic or long division, find the quotient and remainder of the operation* $\frac{3x^3-12x^2+10}{x-2}$.
Here a long division is performed; synthetic division should give the same result:

$$\begin{array}{r|rrrr} & & 3x^2 & -\ 6x & -12 \\ \hline x-2 & 3x^3 & -12x^2 & +\ 0x & +10 \\ & 3x^3 & -\ 6x^2 & & \\ \hline & & -6x^2 & +\ 0x & \\ & & -6x^2 & +12x & \\ \hline & & & -12x & +10 \\ & & & -12x & +24 \\ \hline & & & & -14 \end{array}$$

And thus the quotient is $3x^2 - 6x - 12$, with a remainder of -14. Thus could also be written as

$$\frac{3x^3 - 12x^2 + 10}{x-2} = 3x^2 - 6x - 12 - \frac{14}{x-2}$$

(c) **(2 points)** *Identify all the* potential *rational roots of* $2x^3 - 5x^2 + 2x - 9$. *Do not check which are actual roots.*
By the rational root theorem, numerators of ± 1, ± 3, and ± 9 are possible, and denominators of 1 and 2, for the 12 possibilities:

$$\pm 1, \pm 3 \pm 9, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}$$

5. **(10 points)** *Answer the following questions concerning logarithms.*

(a) **(2 points)** *Express* $5\ln(xy^2) - \frac{1}{4}\ln x - 3\ln(\frac{yz^3}{x})$ *as a single logarithm.*
Using the rule $n \log_b x = \log_b(x^n)$, we can simplify this expression to

$$\ln(xy^2)^5 - \ln x^{1/4} - \ln\left(\frac{yz^3}{x}\right)^3 = \ln(x^5y^{10}) - \ln x^{1/4} - \ln\frac{y^3z^9}{x^3}$$

and then, using the rule $\ln a - \ln b = \ln\frac{a}{b}$ we shall convert this to

$$\ln\frac{x^5y^{10}}{x^{1/4}\frac{y^3z^9}{x^3}} = \ln\frac{x^{31/4}y^7}{z^9}$$

(b) **(3 points)** *Find the domain of the function* $f(x) = \frac{x^2-9}{\log_{10}(x-1)}$.
We have two potential problems: the denominator could be zero, or the argument of the logarithm could be nonpositive. We thus require that $x-1 > 0$ and $\log_{10}(x-1) \neq 0$. The first condition is easily found to be $x > 1$; the second, in contrast, is that $x - 1 \neq 10^0 = 1$, or $x \neq 2$. Thus, the domain consists of values $x > 1$ and $x \neq 2$; in interval notation, x lies in one of the intervals $(1, 2)$ or $(2, +\infty)$.

(c) **(3 points)** *Determine the value of* $\log_5 30 - \frac{1}{2}\log_5 144 + \log_5 10$.
Using our logarithm-simplification rules,

$$\begin{aligned}\log_5 30 - \frac{1}{2}\log_5 144 + \log_5 10 &= \log_5 30 - \log_5\sqrt{144} + \log_5 10 \\ &= \log_5\frac{30\cdot 10}{\sqrt{144}} = \log_5\frac{300}{12} = \log_5 25 = 2\end{aligned}$$

(d) **(2 points)** *Solve for x in the exponential equation $3 \cdot 2^{3x-1} = 48$.*

Isolating the x algebraically, we see:

$$\begin{aligned} 3 \cdot 2^{3x-1} &= 48 \\ 2^{3x-1} &= \frac{48}{3} = 16 \\ 3x - 1 &= \log_2 16 = 4 \\ 3x &= 5 \\ x &= \frac{5}{3} \end{aligned}$$

6. **(10 points)** *Answer the following questions preparatory to sketching the rational function $h(x) = \frac{3(x+2)(x-1)}{x(x+4)}$.*

(a) **(2 points)** *What is the function's domain?*

The denominator cannot be zero, so $x \neq 0$ and $x + 4 \neq 0$, which simplifies to $x \neq 0, -4$. In interval form, this would be $(-\infty, 0), (0, 4), (4, \infty)$.

(b) **(2 points)** *Does this function have x-intercepts, and if so, what are they?*

$h(x) = 0$ when its numerator is zero. $3(x + 2)(x - 1) = 0$ when $x + 2 = 0$ or $x - 1 = 0$. Thus, $h(x)$ has the x-intercepts -2 and 1.

(c) **(2 points)** *Where are this function's vertical asymptotes?*

These are precisely the places where the denominator becomes zero, precipitating a "blow-up" of the function. As seen above, these points are $x = 0$ and $x = 4$.

(d) **(3 points)** *How does this function behave as x becomes very large? How does it behave as x becomes very highly negative? Label which is which.*

This function is, in the long term, behaves much like $\frac{3x^2}{x^2} = 3$. So if we were to look at $h(x)$ at very large values of x, $h(x)$ will be very close to 3; likewise, for x very highly negative, $h(x)$ is close to 3.

(e) **(1 point)** *Does this function have a maximum or minimum value? Why or why not?*

It has neither; we can get arbitrarily large numbers or arbitrarily low numbers by considering x-values close to the asymptotes, e.g. $h(3.999)$ or $h(4.001)$.