1. **(10 points)** *Market research suggests that if tickets to a concert are \$5, there will be 500 tickets sold, but every dollar the price is increased will lose 20 customers.*

   (a) **(3 points)** *Find a function describing the* demand *for tickets as a function of price.*

   Let us denote the ticket price by x; we have a base attendance of 500, and we diminish that by 20 for each dollar by which x exceeds 5; in otherwise, we must take $20(x-5)$ people away from that base attendance to determine the attendance at a ticket price x. Thus our demand function is $D(x) = 500 - 20(x-5)$. This may, but need not, be algebraically simplified to $D(x) = 600 - 20x$.

   (b) **(3 points)** *Find a function describing the total* revenue *from ticket sales as a function of price.*

   As seen above, we sell $600-20x$ tickets at ticket price x; our total revenue from so doing would be the product of the demand and the price, or $R(x) = D(x) \cdot x = (600-20x)x$. This may, but need not, be expanded algebraically to give $R(x) = -20x^2 + 600x$.

   (c) **(4 points)** *Find a sale price for tickets which maximizes revenue, and the total revenue earned at this price. Label which is which.*

   We seek to find the value of x maximizing the quadratic $R(x) = -20x^2+600x$ determined in the previous section; to do so we will locate its vertex, which is most easily done by placing $R(x)$ in standard form:

$$
\begin{aligned}
R(x) &= -20x^2 + 600x \\
&= -20(x^2 - 30x) \\
&= -20(x^2 - 30x + 15^2 - 15^2) \\
&= -20(x^2 - 30x + 15^2) + 20 \cdot 15^2 \\
&= -20(x - 15)^2 + 20 \cdot 15^2
\end{aligned}
$$

   so while k here is a rather unpleasant number ($20 \cdot 15^2 = 4500$), we never actually need to calculate it; we're interested in the x-coordinate of the vertex, which indicates that the minimizing ticket price is \$15.

2. **(15 points)** *Answer the following questions about the functions* $f(x) = \frac{\sqrt{x}}{x-6}$ *and* $g(x) = \log_2(x-3)$. *In each question asking for multiple answers, label which is which.*

   (a) **(3 points)** *Determine the domains of* $f(x)$ *and* $g(x)$.

   Since $f(x)$ has a denominator of $x-6$ and a square-root of x, (but is otherwise free of "dangerous" calculations), we know that the domain of $f(x)$ consists of those points where $x - 6 \neq 0$ and $x \geq 0$, or where $x \neq 6$ and $x \geq 0$; if desired, this can be expressed as the interval form $[0,6) \cup (6,\infty)$.

   On the other hand, $g(x)$ involves a logarithm of $x-3$, but is otherwise free of dangerous calculations, so we know that the domain of $g(x)$ consists of those points where $x-3>0$, or where $x>3$; if desired, this can be expressed as the interval form $(3,\infty)$.

   (b) **(3 points)** *Find the inverse of the function* $g(x)$.

We know that $g(g^{-1}(x)) = x$, and by expanding $g(x)$ we can express $g^{-1}(x)$ in terms of x:

$$\begin{aligned} x &= g(g^{-1}(x)) \\ x &= \log_2(g^{-1}(x) - 3) \\ 2^x &= g^{-1}(x) - 3 \\ 2^x + 3 &= g^{-1}(x) \end{aligned}$$

(c) **(2 points)** *Write formulas, which need not be simplified, for* $(f - g)(x)$ *and* $\frac{f}{g}(x)$.

$$(f - g)(x) = \frac{\sqrt{x}}{x - 6} - \log_2(x - 3)$$

$$\frac{f}{g}(x) = \frac{\frac{\sqrt{x}}{x-6}}{\log_2(x - 3)}$$

(d) **(4 points)** *Determine the domains of* $(f - g)(x)$ *and* $\frac{f}{g}(x)$.

We recall that $f - g$ (and also $f + g$ and fg, althoguh those weren't asked about) has a domain given by the overlap of the domains of f and g, so its domain is given by $x > 3$, $x \geq 0$ and $x \neq 6$; the second of these is actually moot, since $x > 3$ is a stronger condition; if desired, this can be expressed as the interval form $(3, 6) \cup (6, \infty)$.

However, $\frac{f}{g}$ has an additional restriction: in addition to needing x-values in the domain of f and g, $\frac{f}{g}$ is only defined where $g(x) \neq 0$. Since $g(x) = \log(x - 3)$, it is clear that $g(x) = 0$ when $x - 3 = 1$, so we must exclude 4 from the domain of $\frac{f}{g}$; thus $\frac{f}{g}$ has a domain given by the conditions $x > 3$, $x \neq 6$, and $x \neq 4$, or in interal form, $(3, 4) \cup (4, 6) \cup (6, \infty)$.

(e) **(3 points)** *Write formulas, which need not be simplified, for* $f(g(x))$ *and* $g(g(x))$.

$$f(g(x)) = f\big(\log_2(x - 3)\big) = \frac{\sqrt{\log_2(x - 3)}}{\log_2(x - 3) - 6}$$

$$g(g(x)) = g\big(\log_2(x - 3)\big) = \log_2\big(\log_2(x - 3) - 3\big)$$

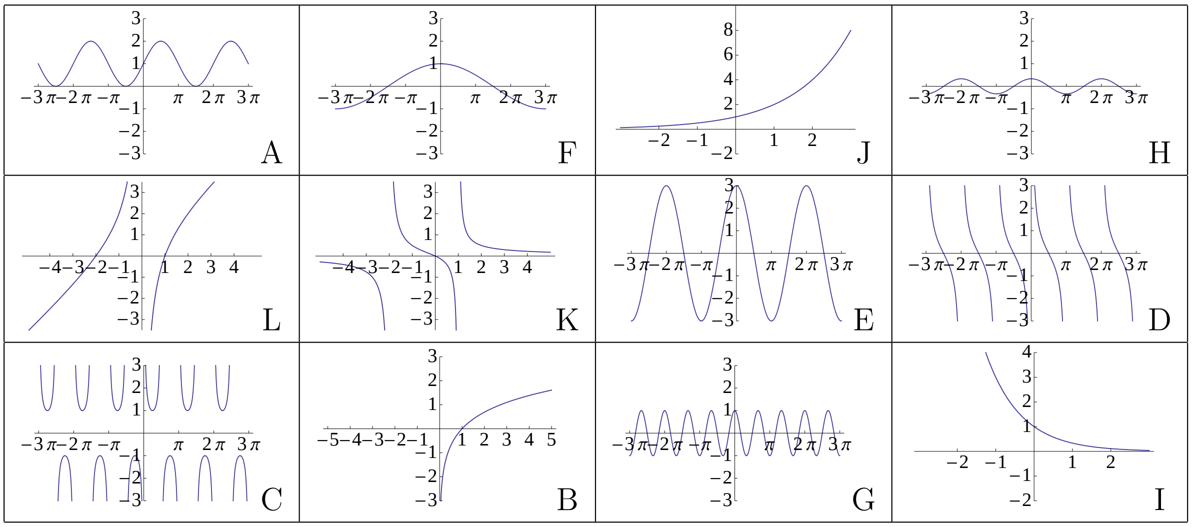
3. **(12 points)** *The following twelve graphs are of the following functions:*

$$A(x) = 1 + \sin x \qquad B(x) = \ln x \qquad C(x) = \csc 2x \qquad D(x) = \cot x$$

$$E(x) = 3\cos x \qquad F(x) = \cos\frac{x}{3} \qquad G(x) = \cos(3x) \qquad H(x) = \frac{1}{3}\cos x$$

$$I(x) = \left(\frac{1}{3}\right)^x \qquad J(x) = 2^x \qquad K(x) = \frac{x}{(x-1)(x+2)} \qquad L(x) = \frac{(x-1)(x+2)}{x}$$

Label each picture with the letter of the appropriate function.



Several of the graphs have highly characteristic and unique shapes which we can identify merely on sight. Graph #10 has an asymptote at 0 and exists only in the positive half of the plane; this is characteristic of a logarithm so it must be $B(x)$. Graph #9 has the disjoint humps and central gap of a secant-like function, so it must be $C(x)$. Graph #9 has the snaky repetition of a tangent-like function, and thus must be $D(x)$.

Of the remainder, we can broadly classify then into sinusoidal, exponential, and "other". Graphs #1, #2, #4, #7 and #11 are sinusoidal and must thus correspond in some order to $A(x)$, $E(x)$, F, and $G(x)$. Graph #1 is vertically shifted, unlike the others, which corresponds to the addition of a constant, as seen in $A(x)$. Graph #2 has period of 6π and graph #11 has period of $\frac{2\pi}{3}$, so they are respectively the result of a horizontal stretch and a compression of the ordinary cosine function: we see that $G(x)$ is a compression and $F(x)$ is a stretch. Both graphs #4 and #7 have the standard period 2π, but they have amplitude $\frac{1}{3}$ and 3 respectively, so they are a vertical compression and stretch of the ordinary cosine function respectively: we see that these will be $H(x)$ and $E(x)$.

Now we handle the exponentials, of which there are two: graph #3 resembles a growth curve while graph #12 is a decay curve. On the functional side, there is $J(x)$, which has a base larger than 1 and thus is a growth curve, while $I(x)$ has a base less than 1 and represents decay.

Finally, the two oddities are graphs #5 and #6, which we recognize as corresponding in some order to the rational functions $K(x)$ and $L(x)$. The easiest visual identification is by vertical asymptotes; $K(x)$ has $(x-1)(x+2)$ in the denominator and must have asymptotes at $x = 1$ and $x = -2$, as graph #6 does, while $L(x)$ has x in the denominator and has an asymptote at $x = 0$, like graph #5.

4. **(13 points)** *Answer the following questions about growth and decay.*

   (a) **(3 points)** *A certificate of deposit (or CD) bought today will increase in value at a relative growth rate of* 1.75% *per year. If you buy a $1000 CD today, how long will it take to reach a value of $1500?*

   The model for the growth of a $1000 investment with a 1.75% rate of growth is $f(t) = 1000e^{0.0175t}$, and here we are asking which value of t yields $f(t) = 1500$. Thus:

$$\begin{aligned} 1000e^{0.0175t} &= 1500 \\ e^{0.0175t} &= \frac{1500}{1000} = \frac{3}{2} \\ 0.0175t &= \ln\frac{3}{2} \\ t = \frac{\ln\frac{3}{2}}{0.0175} &\approx 23.17 \text{ years} \end{aligned}$$

   The approximation step is provided for context, and need not be calculated under exam conditions.

   (b) **(2 points)** *Betaphenethylamine is metabolized and flushed from the bloodstream in such a manner that after one hour, 15% of the drug has been eliminated. Produce a function describing the quantity of the drug still present t hours after administration of a 60mg dose.*

   Each hour, the quantity of beta is reduced to 85% of its previous quantity, so after t hours, it has been reduced to a proportion 0.85^t of its initial quantity. Thus, we may model the decay of this drug with the function $f(t) = 60(0.85^t)$.

   (c) **(3 points)** *Betaphenethylamine users suffer visual hallucinations at levels of 25mg or more. Using the function found in the previous part, determine how long it will take after administration of a 60mg dose for the hallucinations to cease.*

   From the above formula in exponential form, solving for $f(t) = 25$, we get

$$\begin{aligned} 60(0.85^t) &= 25 \\ 0.85^t &= \frac{25}{60} = \frac{5}{12} \\ t &= \log_{0.85}\frac{5}{12} = \frac{\ln\frac{5}{12}}{\ln 0.85} \approx 5.39 \text{ hours} \end{aligned}$$

   The approximation step is provided for context, and need not be calculated under exam conditions.

   (d) **(2 points)** *The population of bacteria in a petri dish doubles every 5 hours. If a colony of the bacteria initially consists of 10 cells, produce a function describing the number of bacteria in the colony after t hours.*

   Since every 5 hours the population doubles, the population will double $\frac{t}{5}$ times in t hours. Thus, the population after t hours is $f(t) = 10 \cdot 2^{t/5}$.

   (e) **(3 points)** *A pot of soup is removed from a hot stove and put in a refrigerator; its temperature in degrees Fahrenheit t minutes after being placed in the fridge is* $f(t) = 45 + 150e^{-0.01t}$. *What is the original temperature of the soup and the temperature of the refrigerator, and how long will it take to cool to* $50°F$*? Label each of your answers.*

The original temperature is definitionally $f(0)$, which can be evaluated to be $45+150e^0 = 45+150 = 195$.

To find when $f(t) = 50$, we solve the equation:

$$\begin{aligned} 45 + 150e^{-0.01t} &= 50 \\ 150e^{-0.01t} &= 5 \\ e^{-0.01t} &= \frac{5}{150} = \frac{1}{30} \\ -0.01t &= \ln\frac{1}{30} \\ t &= \frac{\ln\frac{1}{30}}{-0.01} \approx 340 \text{ minutes} \end{aligned}$$

The approximation step is provided for context, and need not be calculated under exam conditions.

5. **(6 points)** *Calculate the following trigonometric expressions.*

   (a) **(2 points)** $\arccos\frac{-\sqrt{3}}{2}$.
   There are a few familiar cosines, and their associated arccosines should be discernable from the familiarity of their form. Recall that the arccosine function returns values between 0 and π. We know that $\cos\frac{5\pi}{6} = \frac{-\sqrt{3}}{2}$, and thus $\arccos\frac{-\sqrt{3}}{2} = \frac{5\pi}{6}$.

   (b) **(2 points)** $\tan\frac{29\pi}{4}$.
   Since $\frac{29\pi}{4} = 7\pi + \frac{\pi}{4}$ and the tangent function has a period of π, this is simply $\tan\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

   (c) **(2 points)** $\sec\frac{-4\pi}{3}$.
   Since $\frac{-4\pi}{3} = -2\pi + \frac{2\pi}{3}$ and the secant function has a period of 2π, this is simply $\sec\frac{2\pi}{3} = \frac{1}{\cos\frac{2\pi}{3}} = -2$.

6. **(14 points)** *Answer the following questions about sequences and series.*

   (a) **(4 points)** *Identify each of the following sequences as arithmetic, geometric, or neither, and give its common ratio or difference if applicable.*

   - $1, 4, 9, 16, 25, \ldots$.
     Note that the differences of the 4 pairs of consecutive terms seen here are 3, 5, 7, and 9; since these are the not the same, this sequence is not arithmetic. The ratios of the 4 pairs of consecutive terms seen here are 4, $\frac{9}{4}$, $\frac{16}{9}$, and $\frac{25}{16}$; since these are not the same, this sequence is not geometric.
   - $2, 6, 18, 54, 162, \ldots$.
     Note that the differences of the 4 pairs of consecutive terms seen here are 4, 12, 36, and 108; these are not the same, so this sequence is not arithmetic. The ratios of the 4 pairs of consecutive terms seen here, however, are 3, 3, 3, and 3; since these are the same, this sequence is geometric with common ratio 3.
   - $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \ldots$.
     Note that the differences of the 4 pairs of consecutive terms seen here are $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{6}$, and $\frac{1}{6}$; since these are the same, this sequence is arithmetic with common difference $\frac{1}{6}$.

- $2, -5, -12, -19, -26, \ldots$.
  Note that the differences of the 4 pairs of consecutive terms seen here are -7, -7, -7, and -7; since these are the same, this sequence is arithmetic with common difference -7.

(b) **(4 points)** *The fourth term of an arithmetic sequence is 6 and the ninth term is 31. What is the second term of the sequence?*

We set up the template for an arithmetic sequence: $a_n = a_1 + d(n-1)$. Then $6 = a_4 = a_1 + 3d$ and $31 = a_9 = a_1 + 8d$. Thus:

$$\begin{aligned} 6 &= a_1 + 3d \\ 31 &= a_1 + 8d \\ (31-6) &= (a_1+8d) - (a_1+3d) \\ 25 &= 5d \\ 5 &= d \end{aligned}$$

and now we can find $6 = a_1 + 3 \cdot 5$ so $a_1 = 6 - 15 = -9$. Thus, $a_n = -9 + 5(n-1)$.

(c) **(3 points)** *What is the sum of the first 50 terms of the arithmetic sequence* $4, 2, 0, -2, -4, \ldots$*?*

We could determine that, since the first term is 4 and the common difference is -2, then $a_{50} = 4 - 2(49) = -94$, and then use our series-reversal trick:

$$\begin{aligned} S &= \quad 4 + \quad 2 + \quad 0 - \quad 2 - \cdots - 94 \\ S &= -94 - 92 - 90 - 88 - \cdots + \quad 4 \\ 2S &= -90 - 90 - \cdots - 90 = 50(-90) \end{aligned}$$

so $S = \frac{-4500}{2} = -2250$. Alternatively, we could use the formula for the sum of a series:

$$S = a_1 n + \frac{n(n-1)}{2} d = 4 \cdot 50 + \frac{50 \cdot 49}{2}(-2) = 200 - 2450 = -2250$$

(d) **(3 points)** *Leaving at most one unsimplified exponent in your answer, evaluate the geometric partial sum* $5 - 10 + 20 - 40 + 80 - \cdots - 5 \cdot 2^{15}$.

This series is geometric with common ratio -2 and first term 5; we are adding up the first 16 terms (since $5(-2)^{n-1} = -5 \cdot 2^{15}$ when $n - 1 = 15$), so we use the formula:

$$S = \frac{a_1(1-r^n)}{1-r} = \frac{5(1-(-2)^{16})}{1-(-2)} = \frac{5 - 5 \cdot 2^{16}}{3}$$

7. **(10 points)** *Answer the following questions about logarithms.*

(a) **(4 points)** *Calculate the folowing logarithms exactly, giving numerical answers:*

- $\log_3 81$.
  Since $81 = 3^4$, it follows that $\log_3 81 = 4$.
- $\log_5 \frac{1}{25}$.
  Since $\frac{1}{25} = \frac{1}{5^2} = 5^{-2}$, it follows that $\log_5 \frac{1}{25} = -2$.
- $\log_6 1$.
  Since $a^0 = 1$ for all positive a (and specifically for $a = 6$), $\log_6 1 = 0$.

- $\log_4 \frac{1}{8}$.
  Since $\frac{1}{8} = \frac{1}{2^3} = \frac{1}{\sqrt{4}^3} = 4^{-3/2}$, it follows that $\log_5 \frac{1}{25} = \frac{-3}{2}$.

(b) **(3 points)** *Calculate the value of the expression* $\log_3 21 - \log_3 28 + 2\log_3 6$ *exactly.*
Using logarithm-simplification rules:

$$\begin{aligned}\log_3 21 - \log_3 28 + 2\log_3 6 &= \log_3 21 - \log_3 28 + \log_3(6^2)\\ &= \log_3 \frac{21}{28} + \log_3 36\\ &= \log_3 \frac{3 \cdot 36}{4} = \log_3 27\end{aligned}$$

and since $27 = 3^3$, it follows that $\log_3 27 = 3$.

(c) **(3 points)** *Find a value of* x *such that* $4 + 2\log_3 x = 8$.

$$\begin{aligned}4 + 2\log_3 x &= 8\\ 2\log_3 x &= 4\\ \log_3 x &= 2\\ x &= 3^2 = 9\end{aligned}$$

8. **(10 points)** *Answer the following trigonometric questions.*

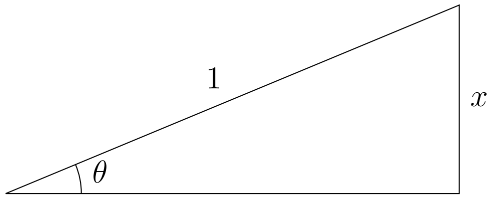
(a) **(3 points)** *If* θ *describes a point in quadrant IV and* $\sin\theta = \frac{-1}{3}$, *what is* $\tan\theta$*?*
We know that $\sin^2\theta + \cos^2\theta = 1$, so:

$$\begin{aligned}\left(\frac{-1}{3}\right)^2 + \cos^2\theta &= 1\\ \frac{1}{9} + \cos^2\theta &= 1\\ \cos^2\theta &= \frac{8}{9}\\ \cos\theta &= \pm\sqrt{\frac{8}{9}} = \frac{\pm\sqrt{8}}{3}\end{aligned}$$

Since θ is in the fourth quadrant, $\cos\theta$ is positive, so $\cos\theta = \frac{\sqrt{8}}{3}$ and then $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{-1}{3}}{\frac{\sqrt{8}}{3}} = \frac{-1}{\sqrt{8}}$.

(b) **(4 points)** *Simplify the expression* $\cot(\arcsin x)$ *to a form which does not use trigonometric functions.*
We give a simple name to $\arcsin x$; traditionally we might call it θ. To convey this information we might build a triangle exemplifying this relationship between θ and x, which we might write as $\theta = \arcsin x$, but more comprehensibly as $\sin\theta = x$; in a right triangle with θ as one of the angles, we know that $\sin\theta$ represents the ratio of the lengths of the opposite side and the hypotenuse. We would thus represent this relationship by

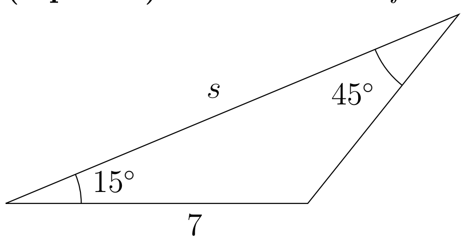
making the opposite side of the triangle have length x, and the hypotenuse have length 1, as shown here:



Furthermore, the adjacent side, which is not labeled in the above picture, can be calculated by the Pythagorean Theorem to have length $\sqrt{1^2 - x^2} = \sqrt{1 - x^2}$.

After that setup, our actual calculation ends up being very easy! We wanted to find $\cot \arcsin x$, which in light of our definition of θ can be written as simply $\cot \theta$. Finding any trigonometric identity of an angle in a labeled right triangle is simply a matter of dividing the appropriate sides: the cotangent is the ratio of the lengths of the adjacent and opposite sides, so in this case, $\cot \theta = \frac{\sqrt{1-x^2}}{x}$.

(c) **(3 points)** *Find the value of s in the triangle (not drawn to scale) below.*



Since the angles add up to 180°, we know that if the unlabeled angle has measure θ, then $45° + 15° + \theta = 180°$, so $\theta = 120°$.

In this triangle all the angles and only one side have been specified, making it a prime candidate for application of the Rule of Sines. Looking at the two labeled sides and their opposite angles, we get:

$$\frac{s}{\sin 120°} = \frac{7}{\sin 45°}$$

which, multiplying both sides by $\sin 120°$, gives $x = \frac{7 \sin 120°}{\sin 45°} = \frac{7 \cdot \frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}} = \frac{7\sqrt{3}}{\sqrt{2}}$ or $\frac{7\sqrt{6}}{2}$.