1. **(12 points)** Let $S = \{\{2,3\}, 2, \{1,2,3,4\}\}$. For each of the following descriptions, either produce a set matching the description or explain briefly why such a set doesn't exist.
   (a) A set A of 4 elements such that $A \subseteq S$.
   (b) A set B of 4 elements such that $B \subseteq \mathcal{P}(S)$.
   (c) A set C of 4 elements, such that $C \in S$.
   (d) A set D of 4 elements, such that $D \in \mathcal{P}(S)$.

2. **(12 points)** Identify each of the following statements as a tautology, a contradiction, or neither. Show your work.
   (a) $(P \wedge Q) \Rightarrow P$.
   (b) $(P \Rightarrow Q) \Leftrightarrow (P \vee Q)$.
   (c) $(P \wedge \neg Q) \wedge (P \wedge Q)$.

3. **(6 points)** Prove or disprove: if n is an integer, then $n^3 - n$ is even.

4. **(6 points)** Prove or disprove: if n is a positive integer, then $3 \mid (2n^2 + 1)$.

5. **(6 points)** Prove that if A, B, and C are sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$.

6. **(6 points)** Let $a_1 = 1$ and for $n > 1$, let $a_n = \sqrt{1 + a_{n-1}}$. Prove that for all n, $1 \leq a_n < 2$.

7. **(12 points)** For each of the following relations R on given sets S, determine whether each of the reflexive, symmetric, and transitive properties hold. Briefly justify your claims.
   (a) $S = \{1,2,3,4,5\}$, $R = \{(1,1),(1,2),(1,3),(2,1),(3,1),(3,3),(4,5),(5,4),(5,5)\}$.
   (b) $S = \mathbb{Z}$, with R given by the criterion that $a \; R \; b$ if and only if $a + b$ is divisible by 3.
   (c) $S = \mathbb{R}$, with R given by the criterion that $a \; R \; b$ if and only if $a - b$ is a non-negative integer.

8. **(6 points)** Prove or disprove: for sets A, B, C, and D, if $|A| = |C|$ and $|B| = |D|$, then $|A \cup B| = |C \cup D|$.

9. **(12 points)** Identify each of the following functions with the stated domains and codomains, as injective, surjective, bijective, or none of the above. Briefly justify your claim.
   (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$.
   (b) $g : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ given by the rule that $g(S)$ is the smallest element of S, with the special-case rule $g(\emptyset) = 1$; e.g. $g(\{3,6,9\}) = 3$.
   (c) $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(n) = n^2$.

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.
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> —Bertrand Russell, *The Study of Mathematics*