1. **(12 points)** *Let $S = \{0, 1, 2, 3\}$. For each of the following descriptions, either produce a set matching the description or explain briefly why such a set doesn't exist.*

   (a) *A set A of 3 elements, such that $A \subseteq \mathcal{P}(S)$.*

   There are actually 56 different correct answers to this question; since $\mathcal{P}(S)$ contains subsets of S, any set containing three subsets of S will suffice. One such example is $A = \{\emptyset, \{1, 2\}, \{0, 1, 2, 3\}\}$.

   (b) *A set B of 3 elements, such that $B \in \mathcal{P}(S)$.*

   There are 4 different correct answers to this question; an element of $\mathcal{P}(S)$, as mentioned above, is a subset of S, so B can be any 3-element subset of S, so for instance $B = \{0, 1, 3\}$ is a satisfactory answer.

   (c) *A set C of 3 elements, such that $C \subseteq S$.*

   The condition $C \subseteq S$ is identical to the condition $C \in \mathcal{P}(S)$, so this question has the same answer (or, as the case may be, 4 possible answers), as the previous question. $C = \{0, 1, 2\}$ is an example of such an answer.

   (d) *A set D of 3 elements, such that $D \in S$.*

   Such a set does not exist: the elements of S are numbers, not sets, so the condition that a set D is an element of S is a nonstarter even without the size requirement.

2. **(12 points)** *For each positive number i, let A_i be the set $\{x \in \mathbb{R} : 1 + \frac{1}{i} \leq x < 2 + \frac{1}{i}\}$, often simply referred to as the half-open interval $\left[1 + \frac{1}{i}, 2 + \frac{1}{i}\right)$. Calculate the following sets:*

   (a) $\bigcap_{i=1}^{3} A_i$.

$$\bigcap_{i=1}^{3} A_i = A_1 \cap A_2 \cap A_3 = [2, 3) \cap \left[\frac{3}{2}, \frac{5}{2}\right) \cap \left[\frac{4}{3}, \frac{7}{3}\right)$$

   The intersection of a finite number of intervals is the interval between the largest lower endpoint and the largest upper endpoint (that is, the place where all three intervals overlap): in this case that overlap is $[2, \frac{7}{3})$.

   (b) $\bigcup_{i=1}^{3} A_i$.

$$\bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3 = [2, 3) \cup \left[\frac{3}{2}, \frac{5}{2}\right) \cup \left[\frac{4}{3}, \frac{7}{3}\right)$$

   The union of a finite number of overlapping intervals is the interval between the smallest lower endpoint and the largest upper endpoint (that is, any point lying in any of the intervals): in this case that overlap is $[\frac{4}{3}, 3)$.

   (c) $\bigcup_{i=1}^{\infty} A_i$.

   We are taking a union of overlapping intervals, so we would expect to get an answer which is itself an interval. The upper edge of the union is easy: A_1 contains values arbitrarily close to and smaller than 3, while none of the subsequent A_i are anywhere near this value, so A_1 will contribute the open upper bound 3. The lower bounds, on the other hand incorporate a new section of the nubmer line for each subsequent value of i:

A_1 has a lower endpoint of 2, but A_2 has a significantly lower bottom edge of $\frac{3}{2}$, while A_3 goes lower still to $\frac{4}{3}$. In fact, numbers arbitrarily close to 1 will lie within some A_i: 1.01 lies in A_{100}, and 1.001 lies in A_{1000}, and so forth. However, 1 itself is not in any A_i, so it will be excluded from the union. Thus this union is $(1, 3)$.

3. **(12 points)** *Write out truth tables for each of the following statements (you may write them all in one truth table, if you wish):*

   (a) $(P \vee Q) \Leftrightarrow Q$.

P	Q	$P \vee Q$	$(P \vee Q) \Leftrightarrow Q$
T	T	T	T
T	F	T	F
F	T	T	T
F	F	F	T

   (b) $P \Rightarrow (P \wedge Q)$.

P	Q	$P \wedge Q$	$P \Rightarrow (P \wedge Q)$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

   (c) $P \vee (\neg P \wedge Q)$.

P	Q	$\neg P$	$(\neg P) \wedge Q$	$P \vee (\neg P \wedge Q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

4. **(6 points)** *Write out the contrapositive of the statement "If x and y are even, then xy is even."*

   We reverse the order of implication and negate each phrase, saying "if xy is not even, then it is not the case that both x and y are even". The latter phrase can be rephrased with DeMorgan's Law: "if xy is not even, then either x is not even or y is not even." If we use the (unstated) assumption that x and y are integers, and thus that anything which is not even is odd, we may further rephrase this statement as "if xy is odd, then either x or y is odd."

5. **(6 points)** *Write the negation of "For all $x \in S$, $f(x) = 0$." as a quantified statement (using a universal or existential quantifier, either in words or in symbols).*

   The statement given is symbolically $\forall x \in S, f(x) = 0$. Its negation, by the rule governing negation of quantifiers, can be written symbolically as $\exists x \in S : f(x) \neq 0$, or "there is an $x \in S$ such that $f(x) \neq 0$."

6. **(12 points)** *Prove the following: for integers a and b, if $a \mid b$, then $a^2 \mid b^2$.*

   Proof. From our premise we may assume $a \mid b$, so $b = ka$ for some integer k. Then $b^2 = (ka)^2 = k^2a^2$. Since k^2 is an integer, it follows that $a^2 \mid b^2$. □

7. **(12 points)** *Prove the following: for an integer n, if $5n - 11$ is even, then n is odd.*

Proof. It is far easier to prove the contrapositive statement: "If n is even, then $5n - 11$ is odd." From our premise we assume n is even, so $n = 2k$ for some integer k. Then $5n - 11 = 5(2k) - 11 = 10k - 11 = 2(5k - 12) + 1$. Since $5k - 12$ is an integer, it follows that $5n - 11$ is odd. $\square$

> A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.
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> —G. H. Hardy, *A Mathematician's Apology*