1 Unions and Intersections

There are two particularly elementary binary operations on sets: union and intersection.

The *union* of two sets A and B, denoted $A \cup B$, is the set consisting of everything that is a member of A or a member of B (or both).

The *intersection* of two sets A and B, denoted $A \cap B$, is the set consisting of everything that is both a member of A and a member of B.

For example: $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$ – note that 1 and 3 don't get repeated, because sets don't contain repetitions! In addition, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$. Things can get trickier when working with infinite sets, of course: we might take $A = \{2, 6, 10, 14, 18, \ldots\}$ and $B = \{3, 8, 13, 18, 23, \ldots\}$. $A \cup B$ doesn't have an easily discernable pattern (although actually it repeats itself in intervals of 20), while $A \cap B = \{18, 38, 58, 78, \ldots\}$.

One case considered noteworthy enough to get its own name is when $A \cap B = \emptyset$; that is, when A and B have no elements in common. Then we call A and B *disjoint*.

Intersections and unions behave quite like many other "nice" operations; you can almost think of unions as quite akin to addition and intersections as being vaguely multiplication-like. The analogy doesn't quite hold up, but note the properties that these operations have:

- $A \cup B = B \cup A$.
- $A \cap B = B \cap A$.
- $A \cup (B \cup C) = (A \cup B) \cup C$.
- $A \cap (B \cap C) = (A \cap B) \cap C$.
- $A \cup \emptyset = A$.
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

We could justify any number of these properties by noting that the membership conditions for the set on the left are the same as the membership conditions for the set on the right.

There is another important operation, and one more quasi-operation. These are all ways to put sets together; what about removing parts of sets? Here we have *set subtraction*. We say that the *difference* of A and B, written $A - B$ or $A \setminus B$, is the set whose elements are those things which are both elements of A and *not* elements of B. Note that this is not commutative: $A - B$ is not $B - A$!

One form which is occasionally useful but problematic is the idea of a *set complement*. Written A^c or $\overline{A}$, the set complement of A consists of all things which are not in A. But here we run into a certain definitional problem: what exactly do we mean by "all things"? This problem is resolved by presuming the existence of a universal set (usually called U) which contains every set under consideration, and then $\overline{A} = U - A$. For instance, if our universe were the reals, $\overline{\mathbb{Q}}$ would be the irrationals.

2 Indexed Sets; Indexed Unions and Intersections

As seen in the last section, one can actually take unions or intersections three at a time without disturbance; liekwise, we could take them four at a time, or five at a time, and so forth. We determine some notation for dealing with these.

When you studied series or Riemann sums, you almost certainly encountered notation such as:

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + \cdots + a_n$$

and might even have seen the less common

$$\prod_{i=0}^{n} a_i = a_0 \cdot a_1 \cdot a_2 \cdots a_n$$

This useful iterated-operation notation can be applied to set operations as well, if we have sets labeled with numeric indices:

$$\cup_{i=0}^{n} S_i = S_0 \cup S_1 \cup S_2 \cup \cdots S_n$$

Applied to finite numbers of sets, this is merely a time-saver, but where it gets interesting is when we apply it to infinite families.

Example: for integers $i > 1$ let us consider $S_i = \{i, 2i, 3i, 4i, \ldots\}$, so that S_i contains every positive multiple of i. Note that any finite union of these sets will leave numbers not divisible by their indices uncovered: e.g. $S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7$ will contain an awful lot of numbers, but it won't contain, say, 11 or 143. But we might note that: $\cup_{i=2}^{\infty} S_i$ contains every positive integer except 1. Why? There are a number of arguments, but the easiest is that $i \in S_i$, so for each i appearing as an index in our big union, i appears in the big union.

Another example: for integers $i \geq 1$, let A_i be the interval $(-\infty, \frac{1}{i})$; alternatively, $A_i = \{x \in \mathbb{R} : x < \frac{1}{i}\}$. Then we can argue that $\cap_{i=1}^{\infty} A_i = (-\infty, 0]$. How? All we need to do is show that nonpositive nubmers are in *every* A_i, and that every positive number is outside of some A_i.

Indices can also be drawn from arbitrary sets.