1 Indexed unions, part 2

Indices can also be drawn from arbitrary sets, so we might refer to the indexed collection $\{A_x\}_{x\in\mathbb{R}}$, given by, for instance, $A_x = \{y \in \mathbb{R} : y is an integer multiple of x\}$, so for instance $A_1 = \mathbb{Z}$, $A_2 = \{\dots, -4, -2, 0, 2, 4, \dots\}$, and $A_\pi = \{\dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots\}$. We could then assert, for instance

$$\bigcap_{x\in\mathbb{R}} A_x = \{0\}$$

or

$$\bigcup_{x\in\mathbb{R}} A_x = \mathbb{R}$$

but we could also look at restricted choiced of indices. For instance, if we were to take $S = \{\frac{1}{3}, \frac{1}{4}, \frac{2}{5}\}$, then

$$\bigcap_{x\in\mathbb{S}} A_x = A_{1/3} \cap A_{1/4} \cap A_{2/5}$$

which would in fact be the set $\{\dots, -4, -2, 0, 2, 4, \dots\}$, and

$$\bigcup_{x\in\mathbb{S}} A_x = A_{1/3} \cup A_{1/4} \cup A_{2/5}$$

which would be a rather complicated mess: $\{\dots, -1, -\frac{4}{5}, -\frac{3}{4}, -\frac{2}{3}, -\frac{1}{2}, -\frac{2}{5}, -\frac{1}{3}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{3}, \dots\}$.

2 Partitions

One particular sort of indexed structure of a set is a *partition*. We say that a collection of distinct nonempty sets $\{A_\alpha\}_{\alpha\in S}$ is a *partition* of A if:

- $\cup_{\alpha\in S} A_\alpha = A$.
- For any two distinct $\alpha, \beta \in S$, A_α and A_β are disjoint.

More informally: a collection of distinct nonempty sets is a partition of A if each element of A appears in *exactly one* set of the partition.

A simple partition: We could partition $\{1, 2, 3\}$ up in 5 different ways: $\{\{1, 2, 3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{2, 3\}, \{1\}\}$, $\{\{1\}, \{2\}, \{3\}\}$.

Another possible partition: we could partition $\mathbb{R}$ into two sets: rationals and irrationals; we could partition $\mathbb{Z}$ into two sets: odds and evens.

A more interesting partition: We could partition $\mathbb{N}$ up into the sets of the form $A_\ell = \{2^n \cdot \ell : n \in \mathbb{Z}, n \geq 0\}$, for odd ℓ: note that, for instance, $A_1 = \{1, 2, 4, 8, 16, \dots\}$, $A_3 = \{3, 6, 12, 24, 48, 96, \dots\}$, $A_5 = \{5, 10, 20, 40, 80, \dots\}$, and since every natural number is uniquely the product of an odd number and a power of 2, every odd number appears in *exactly one* of these sets.

Partitions will prove useful later, when we'll want to prove things which are proved differently for elements of different partitions.

3 Cartesian Products

We don't use these much yet, but they'll be introduced now: this is a way we build sets of *ordered pairs*. The Cartesian product of two sets A and B is a set of all ordered pairs with the first coordinate drawn from A and the second coordinate from B, so

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

You already have encountered one of these long ago: $\mathbb{R} \times \mathbb{R}$ is the coordinate plane.

4 Logic: What is a statement

Now, on to the good stuff! Sets allow us to collect objects (usually numbers) based on the rules they follow. The other major ingredient of mathematical reasoning is *logic*. Logic is the systematization of truth, and how the truth of one statement relates to the truth of another.

First of all, we must ask, what is a statement? A statement is a declaration which can be definitively said to be true or false. Here are ovious statements:

- $2 + 2 = 4$.
- 9 is a prime number.
- All integers are positive.
- There is a Fibonacci number which is also a square number.
- Every even number greater than 2 is the sum of two primes.

Note that not all of these are true. One of them has a truth value which we do not, in fact, know, but we are certain that it is either true or false. Things get a little bit fuzzier if we introduce unknown quantities:

- $x > 2$.
- n is a prime number.
- For all x, $f(x) = f(f(x))$.
- I have my keys in my left pocket.

Depending on what x or f or n is, or where my keys are, any of these might be either true or false, but contextually, they are all guaranteed to be either true or false in any specific context. So these are statements too.

There are several things which are not statements, which are either imperatives, interrogatives, or, in a subtle point, self-referentials:

- Let x be 2.
- Are there any real nubmers x such that $x = f(x)$?
- This statement is false.

We ordinarily give statements names, usually P or Q or subscripted versions thereof. We might let P be "2+2-5", which is a false statement, or let Q be "x is a rational number", which depends on what x happens to be.