1 Logic: Statements concluded

We ordinarily give statements names, usually P or Q or subscripted versions thereof. We might let P be "2+2-5", which is a false statement, or let Q be "x is a rational number", which depends on what x happens to be.

Often, when working with logic, we manipulate just the letters describing a statement, not necessarily considering what the underlying statement is. Thus, we need to account for the prospect that a statemnt might potentially be either *or* false. A helpful tool for enumerating and exploring the possibilities in what is called a *truth table*. When we're just looking at a single statement, the truth table is pretty boring:

P
T
F

but a truth table when considering two statements has to take 4 possible combinations of truth-value into consideration:

P	Q
T	T
T	F
F	T
F	F

and with three statements we would have 8 possible combinations:

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

So far these are purely listing the possible situations, not exploring them. But we don't have much to explore yet — that comes when we combine or midify statements.

2 Logic: the negation operation

One logical operator — a way we can modify statements, is to negate them. For instance, we might negate "5 is prime" to "5 is not prime", or negate "x is a positive integer" to "x is not a positive integer" (note that "x is a negative integer" and "x is a nonpositive integer" won't do).

For a complicated statement, we have to be careful to apply the negation to the outermost level of the statement. So if P is "n has a prime factor which is greater than 5", then not P is "n does not have a prime factor which is greater than 5", as opposed to "n has a prime factor which is not greater than 5". These are different statements! Note that, for instance, they have different truth values for $n = 15$.

The most common way to write the negation of P symbolically is $\neg P$. The book uses $\sim P$, for reasons I will not try to defend or fathom. We can use a truth table to exhibit the relationship of P to $\neg P$:

P	$\neg P$
T	F
F	F

so $\neg P$ is a statement which is false when P is true, and vice versa. We could now note that the negation of the negation of P, written $\neg(\neg P)$, would have the same truth value as P itself.

So, for example, "it is not the case that n is not prime" is a roundabout statement with the same truth or falsity as "n is prime".

3 Logic: the disjunction and conjunction operations

These are binary operators which act on two statements; below we wioll consider possible statements P and Q.

The *disjunction* is the formal name for the operation most people would call "or". The statement "P or Q" is one which is true when at least one of the two statements is true. Symbolically, we write this as $P \vee Q$, and we can see a truth table describing its behavior below:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Thus, for example, the statement "9 is prime or 6 is even" is true, because the sub-proposition "6 is even" is true (even though "9 is prime") is not. On the other hand, "9 is prime or 5 is even" is false, since neither of the sub-propositions is true.

The *conjunction* is a formal name for the other big logical operation, which we ordinarily call "and". The statement "P and Q" is one which is true when both of the two statements are true. Symbolically, we write this as $P \wedge Q$, and we can see a truth table describing its behavior below:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

so, for instance, "9 is prime and 6 is even" is a false statement, because 9 isn't actually prime, but "7 is prime and 6 is even" is true, because both subordinate clauses are true.

We now have enough operators to create extremely involved chains of statements now, and we can use truth tables to explore them. For instance, we might produce a logical statement to describe the proposition "either P and not Q are both true, or at least one of Q and R are true": this would be $(P \wedge \neg Q) \vee (Q \vee R)$. We might be curious about what truth values on P, Q, and R make this true, and which make it false. We can explore this with a truth table, building up the individual subordinate clauses.

P	Q	R	$P \wedge \neq Q$	$Q \vee R$	$(P \wedge \neg Q) \vee (Q \vee R)$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	T	T	T
T	F	F	T	F	T
F	T	T	F	T	T
F	T	F	F	T	T
F	F	T	F	T	T
F	F	F	F	F	F

so by the truth table, we can see that this complicated proposition is true except when all of P, Q, and R are false; in fact, this proposition would be equivalent to "at least one of P, Q, and R is true".

4 Logic: the implication operation

This is the trickiest of the operators conceptually, because it may not mean quite the same as its traditional connotation, but it's a vital one for understanding mathematical statements.

Let us consider the phrases "if P then Q", "Q if P", "P only if Q", "P implies Q" or "when P is true, Q must be true". All three of these describe the same thing, and symbolically they are repesented by $P \Rightarrow Q$. But what does this phrase actually mean? Let us consider a simple statement one might make in everyday life: "if you win this game, then I'll give you five dollars". Considering a situation in which you won and I paid you, I would have kept my promise (i.e. the statement I made would be true). But consider a case where you won and then I gave you nothing — you would be quite justified in saying I had broken my word (i.e. the statement I made was false).

These two cases are pretty straightforward. The interesting cases happen when the premise isn't met. What does "if P then Q" when P is false? Going to our betting example, and the idea of truth as "keeping our word", the answer is obvious: if you don't win the game, then I am keeping my word by not paying you, of course. Furthermore, I would be keeping my word even if I did decide, in a fit of generosity, to give you the money anyways. As a result, "if P then Q" happens to be guaranteed true if P is false, which is not intuitively obvious to most people! Symbolically, we could build the truth table:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

As another example, let's consider the statement "if x is an integer, then $2x$ is also an integer". Hopefully you would agree that this is a true statement, and one we might rewrite as "twice an integer is also an integer". But note that we can verify the universality of this statement by trying our certain numbers x: considering the case $x = 1$, we might have "if 1 is an integer, 2 is also an integer", which probably does not induce any discomfort. But let's try $x = 2.5$ and $x = 1.1$, which would respectively describe the statements "if 2.5 is an integer, then 5 is also an integer" or "if 1.1 is an integer, then 2.2 is also an integer", neither of which seem terribly meaningful as statements, since they are statements premised on a falsehood. But any statement premised on a falsehood is true!

Thus, all manner of statements including truly ridiculous assertions may in fact be true, such as Raymond Smullyan's assertion "if $2 + 2 = 5$, then I am the Pope", which highlights another difference between logical implication and our customary usage: logical implication doesn't require consequentiality, temporal sequentiality, or even any similarity whatsoever between the propositions. "if 3 is an odd number, then the square root of 2 is irrational" is a true statement, but the statement itself doesn't require that $\sqrt{2}$ is irrational *because* 3 is odd.

Some more vocabulary: if $P \Rightarrow Q$, then P is called the *premise* of the implication, and Q the *conclusion*. P might also be called a *sufficient condition* for Q to occure, or Q the *necessary condition* for P to occur.

5 Inverses, Converses, and Contrapositives

There are four closely related statements to "if P then Q" ($P \Rightarrow Q$). One is its *converse* "if Q then P". Note that these are not the same statement! "If I am a vegetarian, then I abstain from pork" is a justifiable assertion, but "If I abstain from pork, then I am a vegetarian" is not. Two other statements related to $P \Rightarrow Q$ are its *inverse* "if not P, then not Q" ($(\neg P) \implies (\neg Q)$) and *contrapositive* "if not Q, then not P" ($(\neg Q) \implies (\neg P)$).

In fact, when we work out the truth values of these four implications, we'll really only have two distinct statements, as seen in the truth table below!

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(\neg P) \Rightarrow (\neg Q)$	$(\neg Q) \Rightarrow (\neg P)$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

so an implication is identical to its contrapositive (i.e. "if I do not abstain from pork, then I am not a vegetarian" is a logically equivalent statement to "if I am a vegetarian, then I abstain from pork"), and furthermore an implication's converse and inverse are identical to each other.