1 Logic: Necessity and sufficiency

Hearken back to the "abstention from pork" example: "if x is a vegetarian, then x abstains from pork". We could state this in terms of the concept of *necessity*, as such:

Abstention from pork is a *necessary* condition of vegetarianism.

which is to say, one can only be a vegetarian if one abstains from pork. Note that the other way around is not true:

Vegetarianism is a *necessary* condition of abstention from pork

since there are several contexts in which one can abstain from pork without being a vegetarian (e.g. kosher or halal diets) — thus, it is not necessary that one be a vegetarian in order to abstain from pork. However, we can phrase the relationship in this way:

Vegetarianism is a *sufficient* condition for abstention from pork

which is to say: if we desire to abstain from pork, we can achieve that through vegetarianism. That's not the only way to achieve it, but it is a *suffuicient* property to ensure it.

Taking it back to logic, if $P \Rightarrow Q$, we call P a *sufficient* condition for Q to occur, and Q a *necessary* condition for P to occur.

We use these in mathematics all the time to characterize various properties. For example, in calculus we might say "continuity is a necessary condition of differentiability", since a function must be continuous to be differentiable (although differentiability requires more than mere continuity, so it is not actually a sufficient condition).

2 Logic: the Biconditional

We saw that a statement and its converse aren't the same, and likewise that a necessary condition isn't the same as a sufficient condition. However, there are many cases where an implication and its converse are both true, such as "if n is an integer, then $2n$ is even, and vice versa" or "if $x \geq 0$, then $\sqrt{x}$ is real, and conversely". Likewise, we may phrase these in necessary and sufficient terms "integrality of n is a necessary and sufficient condition for $2n$ to be even" or "$x \geq 0$ is a necessary and sufficient condition for $\sqrt{x}$ to be real". We can write this concept several ways:

- "If P then Q, and conversely"
- "P implies Q, and conversely"
- "P if and only if Q" (sometimes abbreviated "P iff Q"; this is an informality suitable for notes and problem sets but generally regarded as unacceptable for publication).
- "P is necessary and sufficient for Q"

We also write this statement as $P \Leftrightarrow Q$, and we could specifically define "P iff Q" as "P implies Q and Q implies P", or, symbolically, $P \Leftrightarrow Q \equiv [(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$. We could also define it in terms of its truth table:

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

From this table, we can also see another interpretation of $P \Leftrightarrow Q$: it is actually the statement "P is logically equivalent to Q"; which is to say, a statement asserting P and Q have the same truth value.

3 Tautologies and Contradictions

One interesting product of our "algebra of logic" developed above is that some statements built from named statments will have truth values *not* determined by the truth values of the subsidiary statement. As a simple example, let's look at the statement $P \wedge (\neg P)$ (read: "P or not P"):

P	$\neg P$	$P \wedge \neg P$
T	F	T
F	T	T

so curiously, this statement is true *regardless* of P's truth value! P could be a statment of unknown truth value (like, say, the Riemann Hypothesis), or of indeterminite truth value (like, say, "x is positive"), and this statement would still be true.

An expression built up of named statements which is true regardless of the truth of those individual statements is called a *tautology*. Calling something a tautology or tautological in most forms of math is not a compliment—it suggests the statement is question is vacuous or trivial. But in logic, tautologies are our first glimpse of Universal Truth: for instance, the fact that $P \wedge \neg P$ is a tautology reveals an unsurprising but universal fact: given any statement, either it or its negation is true.

A more complicated tautology: $[(\neg P) \wedge Q] \Leftrightarrow (P \Rightarrow Q)$. This can be shown exhaustively with a truth table. And it too reveals a universal truth: that $[(\neg P) \wedge Q]$ is logically equivalent to $P \Rightarrow Q$.

There is another way a logical expression can be independent of its subsidiary statements; there are *contradictions*, which are false regardless of the truth value of the underlying statements. An example of a contradiction is, for instance, $P \wedge \neg P$; of course, no value of P makes P and $\neg P$ simultaneously true, so $P \wedge \neg P$ is always false.