1 Logic: Quantifiers

We often wish to state that a statement is globally true. For instance, we might say: "for all real x, x^2 is positive". This is what is referred to as a *universally quantified* statement. "Universal" meaning it is an assertion that a statement is true for *every* object satisfying certain conditions (in this case, the condition is that x is a real number). We may express the concept of "for all" symbolically with an upside-down A, which is the symbolic form of the universal quantifier. Thus, the statement "for all real x, x^2 is positive" could be symbolically expressed as:

$$\forall x \in \mathbb{R}, x^2 > 0$$

Incidentally, note that this quantified statement is in fact *false*, because when $x = 0$, the statement $x^2 > 0$ is not true.

Another form of quantifier which is in use is the existential quantifier. Here, instead of asserting that a statement is true for *all* objects satisfying certain conditions, we assert that there is *some* object satisfying those conditions for which the statement is true. So we might, for instance, make an existential assertion like "there is an integer x such that $x^2 = x$". This particular statement is true (there are in fact two, of course: 0 and 1), but we could write it symbolically, using the symbol for the existential quantifier. Read "there exists", we write it as an inverted E symbolically:

$$\exists x \in \mathbb{Z} : x^2 = x$$

A word on usage: I think of a colon as meaning "such that", which is obligatory grammatically to follow "there exists x", while a comma I just read as a comma; others may use commas or colons to separate quantifiers from the quantified statements, depending on their preferences.

We might quantify elements of sets other than the pre-defined ones. So, for instance, if $S = \{1, 2, 3\}$, then the statement $\forall x \in S, x^3 - 6x^2 + 11x = 6$ would be true, since the statement in question is true when $x = 1$, $x = 2$, and when $x = 3$.

There are a number of useful observations to be made about both the trivial cases and specialization of quantifiers:

- The statement $\forall x \in \emptyset, P$ is true regardless of what the statement P is.
- The statement $\exists x \in \emptyset : P$ is false regardless of what the statement P is.

  Both of these are somewhat Zen statements: we are asserting that every element of the empty set has any property we might desire, since there are no elements of the empty set to contradict the claim. Likewise, we are asserting that the empty set does not have an element with a given property, because it does not have an element of any sort.

- The statement $\forall x \in S, P$ implies $\exists x \in S : P$ if S is nonempty.

  This statement is true because, as long as S has at least one element, if *every* element satisfies a property, then surely at least one element satisfies that same property.

- The statement $\neg(\forall x \in S, P)$ is logically equivalent to $\exists x \in S : \neg P$.
- The statement $\neg(\exists x \in S : P)$ is logically equivalent to $\forall x \in S, \neg P$.

  These two are important for representing negations of quantified statements. If we were so consider the logical negation of the universally quantified statement "for all x in S, P is true", it would suffice to say that there is some x in S for which P is not true; in other words, "there exists an x in S such that P is false". Likewise, if we were to negate the existentially quantified "for some x in S, P is true", we are asserting that there is no x making P true; in other words, that "for all x in S, P is false".

We now understand what the quantifiers are and how to manipulate them, but there is one danger to still be watched out for: stacking quantifiers together. In two cases, stacked quantifiers behave the way we might expect:

- We can read $\forall x \in S, \forall y \in T, P$ as "P is true for all x in S and y in T".
- We can read $\exists x \in S : \exists y \in T : P$ as "P is true for some x in S and y in T".

  Sometimes, one will see the second quantifier left out in such expressions, e.g. $\exists x \in S, y \in T : P$; if x and y are drawn from the same set, they may even have their set-membership listed together, e.g. $\exists x, y \in S : P$.

However there are two cases which look similar and are deceptively different:

- We can read $\forall x \in S, \exists y \in T : P$ as "For every x in S, we can choose a y in T which makes P true".
- We can read $\exists y \in T : \forall x \in S, P$ as "We can choose a y in T such that, for all x, P is true".

  It is important to realize that these statements, even though in both of them x is universally quantified and y is existentially quantified, are not the same! The first asserts y can be chosen depending on x, while the second asserts that a single choice of y works for all x. As an exhibition of the difference between these two statements, consider the specific statements:

  $$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x < y$$

  which is a true statement, asserting, in essence, that every real number x is exceeded by some real number y, a true statement (explicitly demonstrable by choosing, say, $y = x + 1$), while the similar statement

  $$\exists y \in \mathbb{R} : \forall x \in \mathbb{R}, x < y$$

  asserts that there is some real number y larger than every real number x, which is emphatically not true (for instance, y is not larger than itself).

You might think double quantification doesn't occur much, but you would think wrong! Consider this doubly-quantified statement:

$$\forall \epsilon > 0, \exists \delta > 0 : \left[(|x - a| < \delta) \Rightarrow (|f(x) - L| < \epsilon)\right]$$

which is identical to a concept you learned in calculus! This is the classic definition of the statement $\lim_{x \to a} f(x) = L$: that no matter how small you demand the distance between $f(x)$ and L is (i.e. "for all choices of that distance ϵ", or $\forall \epsilon > 0$), there is a choice of distance between x and a (i.e. "there exists a choice of that distance δ", or $\exists \delta > 0$), such that if x is within δ of a, then $f(x)$ is within ϵ of L.