1 Logic: Quantifiers Concluded

Just as an illustration of how we can use quantifiers to explain what is going on in a certain context, we might try *negating* the statement from the end of the last class period. So, we might ask, what would it mean, for instance, to say that $\lim_{x\to a} f(x)$ is *not* equal to L? From a symbolic-logic standpoint, this statement is:

$$\neg\left(\forall\epsilon > 0, \exists\delta > 0 : \left[(|x-a| < \delta) \Rightarrow (|f(x)- < \epsilon)\right]\right)$$

On the outermost level, this is a negation of a universally quantified statement, which can be converted into an existentially quantified negated statement:

$$\exists\epsilon > 0 : \neg\left(\exists\delta > 0 : \left[(|x-a| < \delta) \Rightarrow (|f(x) - L| < \epsilon)\right]\right)$$

And now, we have a negated existentially quantified statement, which can be converted into a universally quantified negated statement:

$$\exists\epsilon > 0 : \forall\delta > 0, \neg\left[(|x-a| < \delta) \Rightarrow (|f(x) - L| < \epsilon)\right]$$

Now, we ask: what would the negation of an implication mean? An implication is false only when its premise is true but its consequence false: asserting the negation of an implication is identical to asserting that this does indeed happen, so the above can be converted to:

$$\exists\epsilon > 0 : \forall\delta > 0, (|x-a| < \delta) \wedge \neg(|f(x) - L| < \epsilon)\right]$$

or, simplifying the negation of an arithmetical inequality:

$$\exists\epsilon > 0 : \forall\delta > 0, |x-a| < \delta \wedge |f(x) - L| \geq \epsilon\right]$$

so the assertion $\lim_{x\to a} f(x) \neq L$ could be interpreted logically as stating that there is a positive ϵ such that, no matter what we choose for δ, a value of x within δ of a does not guarantee an $f(x)$ within ϵ of L.

2 Propositions and Proofs: basic form and vacuity

The form of a proposition is almost always "if P, then Q". We often wish to show such a proposition is true in general. One of the confounding things about propositions of the form $P \Rightarrow Q$ is that, if P is false, they are nevertheless true! This serves us well in proof techniques, as it means we don't actually have to concern ourselves at all with the prospect that P might be false — while the falseness of our premise is a possibility, it is not a possibilty which actually affects the truth value of our proposition.

At this point we must dispense with one philosophically troublesome case: what if P is in fact a universally false statement? Such propositions are called *vacuous*, as in these examples

Proposition 1. *If 2 is composite, then there is an integer between 1 and 2.*

Proposition 2. *If* $1 + 1 = 3$*, then* $1 = 2$.

Proposition 3. *If* $x \in \emptyset$*, then* x *is prime*

All three of these are of course true statements, and could be proven merely by appealing to their vacuity: the premise is always false. However, in spite of being true, they aren't *useful* in any real sense: one would never start from the fact that 2 is composite and want to demonstrate, based on this premise, that there is an integer between 1 and 2, because the premise could not have been shown true to begin with. Such propositions are called *vacuous*, because they don't demonstrate any sort of actual relationship.

As we look at simple proof, we'll work from a few defined terms. The next two definitions give us the concept of *parity* of integers:

Definition 1. An integer n is *even* if and only if $n = 2k$ for some integer k.

Definition 2. An integer n is *odd* if and only if $n = 2k + 1$ for some integer k.

One can use a definition as essentially an equivalence between phrases, so that the statements "n is even" and "n is twice some integer k" are considered to be interchangeable. We shall exhibit the use of definitions in our first real proof:

Proposition 4. *If n is an even integer, then $3n - 5$ is odd.*

Proof. By our premise we may assume that n is even, so $n = 2k$ for some integer k. Then, $3n - 5 = 3(2k) - 5 = 6k - 5 = 2(3k - 3) + 1$. Since $3k - 3$ is an integer, $3n - 5$ must be odd. □