1 Direct Proof Concluded

So direct proof has a fundamental strategy: consider the proof in the form "if P then Q". Then, assume P. Proceed via individual results derivable from P and from each other to demonstrate Q. A few more examples, because we learn how to prove things mostly by example:

Proposition 1. *If $n > 2$ is even, then n is composite.*

Proof. Since n is even, $n = 2k$ for some integer k. Furthermore, since $n > 2$, $2k > 2$, from which it follows that $k > 2$. Since $2k$ is thus a product of 2 and an integer greater than 1, $n = 2k$ is composite. □

2 Proof by Contrapositive

Sometimes a statement will be difficult to argue in one directino but seems to have a more natural flow in the other. Consider the statement:

Proposition 2. *For a positive integer n, if n is a prime, then either $n = 2$ or n is odd.*

This is difficult to argue directly (it's not impossible, but it'll require some groundwork), but the proof we just did in the previous section was somewhat akin to this proof in terms of exploring primality and evenness. However, there we argued *from* numeric value and parity to primality, not the other way around.

One easy way is to phrase this in terms of a logically equivalent but differently phrased statement. Recall that the implication "$P \Rightarrow Q$" is logically equivalent to its contrapositive "$(\neg Q) \Rightarrow (\neg P)$". Here, where P is "n is prime" and Q is "either $n = 2$ or n is odd", we could rephrase the implication as:

Proposition 3. *For a positive integer n, if neither n is neither equal to 2 nor odd, then n is not prime.*

which is easy to prove using the exact same argument as before:

Proof. Since n is an integer which is neither equal to 2 not odd, it is specifically the case that $n \neq 2$ and n is odd. Since n is positive and even, $n \geq 2$; fromt he fact that $n \neq 2$, we further know that $n > 2$. By the previous proposition, since $n > 2$ and n is even, n must be composite. □

Note that we used two interesting techniques above: first, the rephrasing of a proposition as its contrapositive, which is the central method of this section, and the invokation of a previous result.

Mathematicians love building new results on old. We could restate the argument in full, but having already said it once, we can simply invoke it later!

This mania for reframing things int erms of already-known situations is summed up by a joke: how would a mathematician, with an empty kettle on a cold stove, boil water? The same way as the rest of us: they turn on the burner, fill the kettle with water, and wait for it to heat up. Now, how would they boil water, with a kettle of cold water on a hot stove. You might just wait for it to heat up — but a mathematician would turn off the stove and empty the kettle: thus reducing the case to "a problem already solved".

Returning to contrapositive methods: this is a common technique used when trying to prove a biconditional. Let us consider the statement:

Proposition 4. *Let n be an integer. $n^2 + 1$ is odd if and only if n is even.*

This is really two propositions in a single statement. We need to prove two different implications, which are converse to each other: "If $n^2 + 1$ is odd, then n is even." and "If n is even, then $n^2 + 1$ is odd". One of these statements is easy using direct methods:

Lemma 1. *Let n be an integer. If n is even, then $n^2 + 1$ is odd.*

Proof. Since n is even, $n = 2k$ for some integer k. Then $n^2 + 1 = (2k)^2 + 1 = 4k^2 + 1 = 2(2k^2) + 1$. Since $2k^2$ is an integer, the expression $2(2k^2)+1$, which is one more than twice an integer, is odd. □

The other direction can be easy, but would be really quite difficult to try to do directly — we'd have to muck around with square roots, and even be certain that $\sqrt{n^2}$ is an integer. We're much happier phrasing it as its contrapositive, which allows our logic to follow a more natural flow from knowledge about n to knowledge about $n^2 + 1$:

Lemma 2. *Let n be an integer. If n is not even, then $n^2 + 1$ is not odd.*

Proof. Since n is an integer, the assumption that n is not even can be rephrased as the assertion that n is odd. Thus $n = 2k+1$ for some integer k, so $n^2+1 = (2k+1)^2+1 = 4k^2+4k+2 = 2(2k^2+2k+1)$, which is even since $2k^2 + 2k + 1$ is an integer. Since $n^2 + 1$ is even, it is not odd. □