1 Proofs with Cases, and Exploitation of Symmetry

We shall now do a proof which looks things we have seen before, but which introduces a novel concept , which is casewise consideration of the premise.

Proposition 1. *Integers x and y are the same parity if and only if $x + y$ is even.*

We'll prove each direction separately, for clarity.

Proposition 2. *If integers x and y have the same parity then $x + y$ is even.*

Proof. There are two ways for x and y to have the same parity, and they must be addressed separately.

Case I: x and y are both even. Then $x = 2k$ for some integer k, and $y = 2\ell$ for some integer ℓ, so $x + y = 2k + 2\ell = 2(k + \ell)$; since $k + \ell$ is an integer, $x + y$ is even.

Case II: x and y are both odd. Then $x = 2k + 1$ for some integer k, and $y = 2\ell + 1$ for some integer ℓ, so $x + y = 2k + 1 + 2\ell + 1 = 2(k + \ell + 1)$; since $k + \ell + 1$ is an integer, $x + y$ is even. □

Note that we divided the premise into two possibilities: the space of values of x and y with the same parity gives rise to two specific possibilities. After a fasihon, we partitioned the set of possible x and y values into two parts and addressed each individually.

The key thing to make certain of, when doing a casewise proof, is that all cases are actually handled. In a simple two-case system like this it's not difficult, but many famous proofs have made use of a great many cases (specifically, both the 1976 Appel-Haken proof of the Four-Color Theorem, whose bulk it devoted to asserting that the thousands of computer-generated cases were in fact exhaustive and Wiles' 1993 proof of Fermat's last theorem, which was in fact *missing* a key case when originally presented and took 2 years to fix).

We also have a casewise argument for the converse.

Proposition 3. *If integers x and y are such that $x + y$ is even, then x and y have the same parity.*

Proof. We shall instead prove the contrapositive, which has the premise that x and y have different parity, and the conclusion that $x + y$ is odd.

There are two ways for x and y to have different parity, and they must be addressed separately.

Case I: x is even and y is odd. Then $x = 2k$ for some integer k, and $y = 2\ell + 1$ for some integer ℓ, so $x + y = 2k + 2\ell + 1 = 2(k + \ell) + 1$; since $k + \ell$ is an integer, $x + y$ is odd.

Case II: x is odd and y is even. Then $x = 2\ell + 1$ for some integer ℓ, and $y = 2k$ for some integer k, so $x + y = 2\ell + 1 + 2k = 2(k + \ell) + 1$; since $k + \ell$ is an integer, $x + y$ is odd. □

You might have noticed that case II was awfully similar to case I there, to the extent that we could have simply appealed to symmetry and handwave through saying "this is identical to case I, except with x and y's expansions reversed". In fact, using a clever and quite common exploitation of symmetry, we can prove the above proposition without cases at all:

Proposition 4. *If integers x and y are such that $x + y$ is even, then x and y have the same parity.*

Proof. We shall instead prove the contrapositive, which has the premise that x and y have different parity, and the conclusion that $x + y$ is odd.

If x and y have different parity, then one of them is even and one of them is odd. Without loss of generality, we may presume x is even and y is odd. Then $x = 2k$ for some integer k, and $y = 2\ell + 1$ for some integer ℓ, so $x + y = 2k + 2\ell + 1 = 2(k + \ell) + 1$; since $k + \ell$ is an integer, $x + y$ is odd. □

The key clever phrase there was "without loss of generality". It was an admission that x and y serve interchangable roles in the problem statement, and thus when a property (i.e. being even) belongs to one of them, it can be assigned arbitrarily to a particular one (i.e. x), with confidence that the same argument would prevail if we were attach that property to the name y instead.

Below is an extensive and attractive proof, which makes use of both casewise argumentation *and* invokes a symmetry-simplifying specification as above.

Theorem 1. *If every one of the fifteen segments between vertices of a regular hexagon is colored red or blue, then three of the hexagons vertices are joined by segments of the same color.*

Proof. Let us call the vertices of the hexagon $x_1, x_2, x_3, x_4, x_5, x_6$, and consider the five segments between x_1 and each of the other five. If we color each of these 5 segments red or blue, it is certain that 3 of the segments are the same color (since the possible color-distributions among the 5 segments are: 5 the same color, 4 one color and 1 the other color, or 3 one color and 2 the other color; no matter which distribution we choose, 3 segments are guaranteed to be the same color). Without loss of generality, we may call the color of these three segments "red". Thus, x_1 is joined to some x_a, x_b, and x_c by red segments. Now, we shall consider the possible colors of the segments among x_a, x_b, and x_c:

Case I: The segment between x_a and x_b is red. Then, the three points x_1, x_a, and x_b are mutually joined by red segments.

Case II: The segment between x_a and x_c is red. Then, the three points x_1, x_a, and x_c are mutually joined by red segments.

Case III: The segment between x_b and x_c is red. Then, the three points x_1, x_b, and x_c are mutually joined by red segments.

Case IV: None of the segments among x_a, x_b, and x_c are red. Then all three of them are blue, so the three points x_a, x_b, and x_c are mutually joined by blue segments.

□

One could simplify this argument by reducing cases I–III into a single case.

A nongeometric interpretation of this fact: if 6 people get together, then either three of them are mutually acquainted or three of them are mutually strangers. This follows from associating each person with a vertex, and coloring the segment between two people red or blue depending on their acquaintance.