1 Recursive definition, introduced

A concept closely related to induction is *recursion*, encountered most often in a sequence-building construct called a *recurrence relation*:

Definition 1. A *recurrence relation of order r* consists of several definitions for the terms of a sequence: certain low-numbered terms $a_1, a_2, \ldots, a_r$ are defined, and for all $n > r$, the equation $a_n = f(a_{n-1}, a_{n-2}, a_{n-3}, \ldots, a_{n-r})$ holds for some given function f.

For instance, one might specify a simple recurrence relation (of order 1) as such; we define a sequence a_n:

$$\begin{aligned} a_1 &= 1 \\ a_n &= 3a_{n-1} - 1 \text{ for } n > 1 \end{aligned}$$

and we can calculate individual values in the sequence by making use of previous values:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 3a_1 - 1 = 3 \cdot 1 - 1 = 2 \\ a_3 &= 3a_2 - 1 = 3 \cdot 2 - 1 = 5 \\ a_4 &= 3a_3 - 1 = 3 \cdot 5 - 1 = 14 \\ a_5 &= 3a_4 - 1 = 3 \cdot 14 - 1 = 41 \end{aligned}$$

Note that this is to computation what induction is to logic: we derive each new term based on our knowledge of the previous term. This means that recursively calculating a_{100}, for instance, would be quite difficult; we'd need to first work out every predecessor term a_{99}, a_{98}, etc. But since induction serves as a proof method wihich is a counterpart to recursion as a computational method, we'll see induction is useful for proving closed formulas for recurrences, when we can find them.

The pattern may not be obvious looking at the values of a_n; But looking at $2a_n$, we see numbers very close to powers of 2, which might lead us to a hypothesis.

Conjecture 1. *For all $n \geq 1$, $a_n = \frac{3^{n-1}+1}{2}$.*

Proving this directly would be tricky (not impossible, but a real pain), since each a_n doesn't have an immediately calculable definition, but appeals to the value of a_{n-1}. If we could assume a_{n-1} took on a particular value, we could appriach a proof much more effectively: thus this is a good candidate for induction, since assuming a case k helps us immensely in exploring the case $k+1$!

Proposition 1. *For all $n \geq 1$, $a_n = \frac{3^{n-1}+1}{2}$.*

Proof. We shall prove this by induction on n. The base case $n = 1$ is easily verified arithmetically; the recurrence states $a_1 = 1$, and $\frac{3^{1-1}+1}{2} = \frac{2}{2} = 1$.

Now, for the induction step, we will fix an integer value $k \geq 1$ and assume the inductive hypothesis $a_k = \frac{3^{k-1}+1}{2}$; armed with this assumption, we will calculate a_{k+1}:

$$a_{k+1} = 3a_k - 1 = 3 \cdot \frac{3^{k-1}+1}{2} - 1 = \frac{3^k+3}{2} - 1 = \frac{3^k+1}{2}$$

Thus $a_{k+1} = \frac{3^{(k+1)-1}+1}{2}$, completing our inductive step. □

Similar methods will verify any true conjecture we would make about any *first-order* recurrence. But what about higher orders? There we need a stronger inductive method. Recall that we had a basic form of induction:

Theorem 1 (Mathematical Induction)**.** *Let $P(n)$ be a statement qualified by a positive integer n. If $P(1)$ is true, and if $P(k)$ implies $P(k+1)$ for all positive integers k, then $P(n)$ is true for all positive integers n.*

Theorem 2 (Mathematical Induction with arbitrary base case)**.** *Let $P(n)$ be a statement qualified by an integer n. If $P(n_0)$ is true for a specific n_0, and if $P(k)$ implies $P(k+1)$ for all $k \geq n_0$, then $P(n)$ is true for all integers $n \geq n_0$.*

Theorem 3 (Strong Mathematical Induction with arbitrary base case)**.** *Let $P(n)$ be a statement qualified by an integer n. If $P(n_0)$ is true for a specific n_0, and if $(P(n_0) \wedge P(2) \wedge \cdots \wedge P(k))$ implies $P(k+1)$ for all $k \geq n_0$, then $P(n)$ is true for all integers $n \geq n_0$.*

This stronger form of induction means that in our inductive step we are not limited to only assuming $P(k)$; if we want to, we can assume every single qualified statement from our base case up to statement k. This is useful for exploring recurrences of order greater than 1; now we can take for granted the computation not only of a_k, but also of a_{k-1}, and a_{k-2}, and so forth.

Let's look at a new recurrence, of order 2:

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 7 \\ a_n &= 7a_{n-1} - 12a_{n-2} \text{ for } n > 2 \end{aligned}$$

The pattern for this one is quite likely not obvious at all to you, but let's skip to the solution:

Proposition 2. *For a_n given by the recurrence above, $a_n = 3^n + 4^n$ for all positive integers n.*

Proof. We prove this result by induction on n. Our base cases $n = 1$ and $n = 2$ are easily demonstrated: $a_1 = 2$ from the recurrence, and $3^0 + 4^0$ is also 2; likewise $a_2 = 7 = 3^1 + 4^1$.

For our inductive step, we fix a $k \geq 2$, and then we may assert that $a_n = 3^n + 4^n$ for all integers n such that $1 \leq n \leq k$. In particular, we may assert that $a_{k-1} = 3^{k-1} + 4^{k-1}$ and $a_k = 3^k + 4^k$. Armed with these hypothesis, we may calculate a_k:

$$\begin{aligned} a_{k+1} &= 7a_k - 12a_{k-1} \\ &= 7\left(3^k + 4^k\right) - 12\left(3^{k-1} + 4^{k-1}\right) \\ &= 7 \cdot 3^k + 7 \cdot 4^k - 4 \cdot 3 \cdot 3^{k-1} - 3 \cdot 4 \cdot 4^{k-1} \\ &= (7-4)3^k + (7-3)4^k \qquad = 3^{k+1} + 4^{k+1} \end{aligned}$$

completing our inductive step. ∎

Even when we don't have an explicit formula for a recurrence, we can often work out valuable information about it using inductive methods:

Proposition 3. *Let b_n be given by a recurrence relation: $b_1 = 1$, $b_2 = 2$, and $b_n = nb_{n-1} + b_{n-2}$ for $n > 2$. For positive integers n, $2 \mid b_n$ if and only if $2 \mid n$.*

Proof. Let $P(n)$ be the assertion that $2 \mid b_n$ if and only if $2 \mid n$; alternatively, it could be phrased as the assertion that b_n and n have the same parity. The base cases $P(1)$ and $P(2)$ are easily verified: $P(1)$ is odd as is 1, and $P(2)$ is even as is 2.

Now let us fix a $k \geq 2$, and inductively assume that $P(1)$, $P(2)$, $\ldots$, $P(k)$ are all true. Specifically, we will use the facts that $P(k)$ and $P(k-1)$ are true below. Now we have two cases: either $2 \mid (k+1)$, in which case our goal is to show that $2 \mid b_{k+1}$, or $2 \nmid (k+1)$, in which case our goal is to show that $2 \nmid b_{k+1}$. We address these casewise:

Case I: $2 \mid k+1$ **(a.k.a.** $k+1$ **is even).** Since $k+1$ is even, k is odd and $k-1$ is even, and since we presumed $P(k)$ and $P(k-1)$ to be true, it thus respectively follows that b_k is odd and b_{k-1} is even. Then, we note that $b_{k+1} = (k+1)b_k + b_{k-1}$; this is a sum of a product of an even and odd number with an even number; the result of such a computation is even, so $2 \mid b_{k+1}$.

Case II: $2 \nmid k+1$ **(a.k.a.** $k+1$ **is odd).** Since $k+1$ is odd, k is even and $k-1$ is odd, and since we presumed $P(k)$ and $P(k-1)$ to be true, it thus respectively follows that b_k is even and b_{k-1} is odd. Then, we note that $b_{k+1} = (k+1)b_k + b_{k-1}$; this is a sum of a product of an odd and even number with an odd number; the result of such a computation is odd, so $2 \nmid b_{k+1}$. □