Even when we don't have an explicit formula for a recurrence, we can often work out valuable information about it using inductive methods:

Proposition 1. *Let b_n be given by a recurrence relation: $b_1 = 1$, $b_2 = 2$, and $b_n = nb_{n-1} + b_{n-2}$ for $n > 2$. For positive integers n, $2 \mid b_n$ if and only if $2 \mid n$.*

Proof. Let $P(n)$ be the assertion that $2 \mid b_n$ if and only if $2 \mid n$; alternatively, it could be phrased as the assertion that b_n and n have the same parity. The base cases $P(1)$ and $P(2)$ are easily verified: $P(1)$ is odd as is 1, and $P(2)$ is even as is 2.

Now let us fix a $k \geq 2$, and inductively assume that $P(1)$, $P(2)$, ..., $P(k)$ are all true. Specifically, we will use the facts that $P(k)$ and $P(k-1)$ are true below. Now we have two cases: either $2 \mid (k+1)$, in which case our goal is to show that $2 \mid b_{k+1}$, or $2 \nmid (k+1)$, in which case our goal is to show that $2 \nmid b_{k+1}$. We address these casewise:

Case I: $2 \mid k+1$ (a.k.a. $k+1$ is even). Since $k+1$ is even, k is odd and $k-1$ is even, and since we presumed $P(k)$ and $P(k-1)$ to be true, it thus respectively follows that b_k is odd and b_{k-1} is even. Then, we note that $b_{k+1} = (k+1)b_k + b_{k-1}$; this is a sum of a product of an even and odd number with an even number; the result of such a computation is even, so $2 \mid b_{k+1}$.

Case II: $2 \nmid k+1$ (a.k.a. $k+1$ is odd). Since $k+1$ is odd, k is even and $k-1$ is odd, and since we presumed $P(k)$ and $P(k-1)$ to be true, it thus respectively follows that b_k is even and b_{k-1} is odd. Then, we note that $b_{k+1} = (k+1)b_k + b_{k-1}$; this is a sum of a product of an odd and even number with an odd number; the result of such a computation is odd, so $2 \nmid b_{k+1}$. □

We can try this with an even more complicated recurrence. Here's one which doesn't have a specific order, but *is* a recurrence relation, and is actually a quite significant set of numbers:

Definition 1. The *Catalan numbers* $C_1, C_2, C_3, \ldots$ are given by the recurrence relation:

$$C_1 = 1$$
$$C_n = \sum_{i=1}^{n-1} C_i C_{n-i} \text{ for } n > 1$$

so, for example, we can work out the first few Catalan numbers:

$$C_1 = 1$$
$$C_2 = C_1C_1 = 1 \cdot 1 = 1$$
$$C_3 = C_2C_1 + C_1C_2 = 1 \cdot 1 + 1 \cdot 1 = 2$$
$$C_4 = C_3C_1 + C_2C_2 + C_1C_3 = 2 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 = 5$$
$$C_6 = C_4C_1 + C_3C_2 + C_2C_3 + C_1C_4 = 5 \cdot 1 + 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 5 = 14$$

and the next few are 42, 132, 429, 1430.

We can prove a couple of things about the Catalan numbers. For instance, their growth rate appears to be quite fast, which we can quantify pretty painlessly by showing that it's at least exponential:

Proposition 2. *For every positive integer n, $C_n \geq 2^{n-2}$.*

Proof. We shall seperately demonstrate the truth of this statement for $n = 1$, since $C_1 = 1 \geq 2^{-1}$; henceforth we proceed by induction with the base case $n = 2$. We prove the base case arithmetically: $C_2 = 1 = 2^0$, so this inequality holds when $n = 2$. We now proceed to the inductive step.

We fix a k, and may then accept the hypothesis that $C_i \geq 2^{i-2}$ for all $i \leq k$. We now want to palce a lower bound on the value of C_{k+1}, which we can do since C_{k+1} is calculated based on the values of various smaller C_i.

$$\begin{aligned} C_{k+1} &= C_1C_k + C_2C_{k+1} + \cdots + C_kC_1 \\ &\geq C_1C_k + C_kC_1 \\ &\geq 1 \cdot C_k + C_k \cdot 1 \\ &\geq 2C_k \geq 2(2^{k-2}) = 2^{(k+1)-2} \end{aligned}$$

□

In actuality even better upper bounds on the Catalan numbers are possible.

Playing with these further, we might discover another neat property of Catalan numbers:

Conjecture 1. *For a positive integer n, C_n is odd if and only if n is a power of 2.*

Proving this is actually quite easy, with the use of a helpful lemma.

Lemma 1. *For every positive integer n, C_{2n} has the same parity as C_n.*

and then we can prove our conjecture easily by use of this fact.

Proposition 3. *For every positive integer n, C_n is even if and only if n is a power of* 2.

Proof. For clarity, let us restate exactly what we want to prove for each positive integer n: we wish to show that if n is a power of 2, then C_n is odd, and otherwise, C_n is even. We shall prove this by induction on n. The cases $n = 1$ and $n = 2$ are obviously true since C_1 and C_2 are both odd (and 1 and 2 are powers of 2).

Fixing $k \geq 2$, we may assume the inductive hypothesis that the parities of each C_i for $1 \leq i \leq k$ follows the observed pattern. Now, we must divide into cases:

Case I: $k + 1$ is a power of 2. Since $k + 1 > 2$ and $k + 1 = 2^r$ for some positive integer r, it is easy to see that $r > 1$ and thus $k + 1 = 2\left(2^{r-1}\right)$. By our lemma, C_{k+1} and $C_{2^{r-1}}$ have the same parity, and by our inductive hypothesis, since 2^{r-1} is a power of 2, $C_{2^{r-1}}$ is odd.

Case I: $k+1$ is an even number which is not a power of 2. Since $k+1$ is even and positive, $k + 1 = 2\ell$ for some positive integer ℓ. By our lemma, C_{k+1} and C_ℓ have the same parity. Since $k + 1$ is not a power of 2, ℓ is not a power of 2, since if it were the case that ℓ were a power of 2, so would $k + 1$ be. Thus, by our inductive hypothesis, C_ℓ is even, so C_{k+1} is even.

Case III: $k + 1$ is an odd number greater than 2 (and thus not a power of 2). Then $k + 1 = 2\ell + 1$ for some integer ℓ, and the recurrence for C_{k+1} is:

$$C_{k+1} = C_1C_k + C_2C_{k-1} + \cdots + C_\ell C_{\ell+1} + C_{\ell+1}C_\ell + \cdots + C_{k-1}C_2 + C_kC_1$$

which can be rearranged into $C_{k+1} = 2\left(C_1C_k + C_2C_{k-1} + \cdots + C_\ell C_{\ell+1}\right)$, so C_{k+1} is even. □

1 Induction and finite sets

One can prove things by induction on finite sets. For instance, take the problem on the first pset.

Give that old saw about how all horses are the same color, unless there's a volunteer.