Using the lemma from last week:

Lemma 1. *For every positive integer n, C_{2n} has the same parity as C_n.*

and then we can prove our conjecture easily by use of this fact.

Proposition 1. *For every positive integer n, C_n is even if and only if n is a power of 2.*

Proof. For clarity, let us restate exactly what we want to prove for each positive integer n: we wish to show that if n is a power of 2, then C_n is odd, and otherwise, C_n is even. We shall prove this by induction on n. The cases $n = 1$ and $n = 2$ are obviously true since C_1 and C_2 are both odd (and 1 and 2 are powers of 2).

Fixing $k \geq 2$, we may assume the inductive hypothesis that the parities of each C_i for $1 \leq i \leq k$ follows the observed pattern. Now, we must divide into cases:

Case I: $k+1$ is a power of 2. Since $k + 1 > 2$ and $k + 1 = 2^r$ for some positive integer r, it is easy to see that $r > 1$ and thus $k + 1 = 2\left(2^{r-1}\right)$. By our lemma, C_{k+1} and $C_{2^{r-1}}$ have the same parity, and by our inductive hypothesis, since 2^{r-1} is a power of 2, $C_{2^{r-1}}$ is odd.

Case I: $k+1$ is an even number which is not a power of 2. Since $k+1$ is even and positive, $k + 1 = 2\ell$ for some positive integer ℓ. By our lemma, C_{k+1} and C_ℓ have the same parity. Since $k + 1$ is not a power of 2, ℓ is not a power of 2, since if it were the case that ℓ were a power of 2, so would $k + 1$ be. Thus, by our inductive hypothesis, C_ℓ is even, so C_{k+1} is even.

Case III: $k+1$ is an odd number greater than 2 (and thus not a power of 2). Then $k + 1 = 2\ell + 1$ for some integer ℓ, and the recurrence for C_{k+1} is:

$$C_{k+1} = C_1C_k + C_2C_{k-1} + \cdots + C_\ell C_{\ell+1} + C_{\ell+1}C_\ell + \cdots + C_{k-1}C_2 + C_kC_1$$

which can be rearranged into $C_{k+1} = 2\left(C_1C_k + C_2C_{k-1} + \cdots + C_\ell C_{\ell+1}\right)$, so C_{k+1} is even. □

1 Induction and finite sets

One can prove things by induction on finite sets. For instance, take the problem on the first pset: why does a set with n elements have 2^n subsets?

Give that old saw about how all horses are the same color, unless there's a volunteer.

2 Relations

We define a *relation on a set S* as a subset of $S \times S$. Note that there are $2^{|S|^2}$ possible relations, including a null relation and an omnipresent relation. Discuss the notation xRy, then define reflexivity, symmetry, and transitivity. There are 8 possible combinations of these properties, and examples can be found for most of them. If time permits, discuss equivalences.