1 Explorations of properties of specific relations

Let's look at some specific relations and demonstrate that they have certain properties. Fir instance, we might consider the following example:

For sets A and B, $A \sim B$ if $A \cap B = \emptyset$.

We might test various properties of $\sim$ Reflexivity, for instance, is easily denied: if S is a set, then $S \cap S = S$, which is, in general, *not* $\emptyset$, so it will not generally be true that $S \sim S$.

Symmetry is true: if A and B are sets, then $A \cap B = B \cap A$, so either both are the empty set or neither are.

Transitivity is not true: consider, e.g. $A = \{1, 2\}$, $B = \{3\}$, and $C = \{1\}$: then $A \sim B$ and $B \sim C$ but $A \not\sim C$.

Let's try another relation, for instance on the reals: $x \sim y$ if $|x - y| \leq 1$. This is reflexive (since $|x - x| \leq 1$), symmetric (since $|x - y| = |y - x|$, so both differences are either greater than 1 or not). This is not transitive (maybe someone has a simple example).

Finally, we might look at $x \sim y$ if x and y are the same parity. This has all three properties, making it a particularly extraordinary type or relation called an *equivalence relation*.

First though, let's look at some manipulations of relations, since you were asked at length to manipulate them:

Definition 1. We might define the *complement* $\overline{R}$ of a relation R on a set S to be $S \times S - R$.

We might now ask how R's properties relate to those of $\overline{R}$. Several proofs are easy:

Proposition 1. *If R is reflexive and $S \neq 0$, then $\overline{R}$ is not reflexive.*

Proof. Since $S \neq 0$, there is some $x \in S$, so by reflexivity, $(x, x) \in R$. Thus $(x, x) \notin \overline{R}$, so x is not related to x by $\overline{R}$, so $\overline{R}$ is not reflexive. □

Proposition 2. *If R is symmetric, then $\overline{R}$ is symmetric.*

Proof. Consider some $(a, b) \in \overline{R}$. By definition $(a, b) \notin R$. Since R is symmetric, it is thus necessary that $(b, a) \notin R$ (since if $(b, a) \in R$, symmetry would require the already-refuted statement that $(a, b) \in R$ be true). Since $(b, a) \notin R$, it follows that $(b, a) \in \overline{R}$. □

There is no good definite statement to be made about transitivity. Some investigation will show that some transitive relations have transitive complements, while some will not.

2 Equivalence relations

Definition 2. A relation which is reflexive, symmetric, and transitive is known as an *equivalence relation.*

These are pretty useful because they provide the vital information that everything is related to itself, the order of relation doesn't matter, and that everything related to a particular element is related to each other. We shall see that equivalence relations end up having a vital relationship with set partitions.

Consider the relationship on the integers a while back, where $a \sim b$ if a and b have the same parity. Then, every even nubmer relates to every even number, and every odd number relates to every other odd number. In a real sense, this is associated with a partition of the integers into an "odd" set and an "even" set, with all the numbers lying in a specific set related to each other.

We formalize this idea with a defnition which could be applied to any equivalence relation:

Definition 3. The *equivalence class*, denoted $[a]$, of a with respect to an equivalence relation R on a set S, is the set $\{b \in S : aRb\}$.

so, that is to say, the set of all b to which a is related. In theory we could define this set for non-equivalence classes, but here is a remarkable fact which is true only for equivalence classes.

Theorem 1. *For an equivalence relation R on a set S, with $a, b \in S$, the following statements are true:*

- $a \in [a]$,
- *if aRb, then $[a] = [b]$,*
- *if it is not the case that aRb, then $[a] \cap [b] = \emptyset$*

Proof. Our first statement is trivial: by reflexivity, aRa, so by the definition of the equivalence class, $a \in [a]$.

Our second statement will be proven by first showing that $[a] \subseteq [b]$, and then $[b] \subseteq [a]$. Consider some $x \in [a]$; by definition of the equivalence class, this means aRx, so by symmetry, xRa. The presime of the statement to be proven asserts that aRb, so by transitivity, xRb and then by symmetry bRx, so $x \in [b]$. An extremely similar argument serves to show that if $y \in b$, then bRy, which together with aRb and transitivity yields aRy so $y \in [a]$. Thus every element of $[a]$ is in $[b]$ and vice versa, so $[a] = [b]$.

For our third statement, we proceed by contradiction, so suppose $(a, b) \notin R$ and that $[a] \cap [b]$ is not empty, so there is some $x \in [a] \cap [b]$, so aRx and bRx. Then, by symmetry, xRb, so by transitivity, aRb, contradicting our premise. □

Corollary 1. *The equivalence classes partition S.*

Proposition 3. *Since $a \in [a]$, all equivalence classes are nonempty, and every element of S lies in some equivalence class. Since nonequal equivalence classes are disjoint, the distinct equivalence classes are disjoint sets whose collective union contains every element of S, so they partition S.*

What may be less obvious is that any partition of S uniquely determines an equivalence relation: if we have partition of S into disjoint sets A_i, we could define a relation on S by stating that $a \sim b$ if a and b lie in the same part A_i.

3 Congruence

Back when we were originally discussing congruence, we proved that for any integers n and a, $a \equiv a \pmod{n}$; likewise, if $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$, and if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$. Thus congruence modulo n is an equivalence relation, and as such gives rise to *congruence classes*, which are a beloved element of number theory and algebra.

One theorem (which is nontrivial and won't be shown here, is that for any positive integer n, an integer k can always be written as $qn + r$, where $0 \leq r < n$. In consequence, we know every k is congruent to a number from 0 to n, so conventionally we label the *congruence classes modulo* n with:

$$\begin{aligned}
[0] &= \{\ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots\} \\
[1] &= \{\ldots, 1-3n, 1-2n, 1-n, 1, 1+n, 1+2n, 1+3n, \ldots\} \\
[2] &= \{\ldots, 2-3n, 2-2n, 2-n, 2, 2+n, 2+2n, 2+3n, \ldots\} \\
&\vdots \\
[n-1] &= \{\ldots, -2n-1, -n-1, -1, n-1, 2n-1, 3n-1, 4n-1, \ldots\}
\end{aligned}$$