1 Function Attributes and Properties

If a function is defined as $f : A \to B$, then the set A is called the *domain* of f; the set B is the *codomain* of f.

We might be curious about which elements of B are actually achieved by f; i.e., which $y \in B$ are such that for some x in A, $f(x) = y$? We might call this set the *image* or *range*. Put in glorious set-builder-and-logic notation, the image of a function $f : A \to B$ is

$$\{y \in B : \exists x \in A, f(x) = y\}$$

We might look at a simple example: consider a function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3\}$ such that $f(1) = 2$, $f(2) = 3$, $f(3) = 3$, $f(4) = 2$, and $f(5) = 2$. Alternatively, we might describe f as $\{(1, 2), (2, 3), (3, 3), (4, 2), (5, 2)\}$. Then, as stated in the declaration of f, f has domain $\{1, 2, 3, 4, 5\}$ and codomain $\{1, 2, 3\}$. The image of f is the set of second co-ordinates actually appearing in f, which is $\{2, 3\}$ — note that this is not quite the entire codomain.

Here's a more complicated example. We might let $g : \mathbb{N} \to \mathcal{P}(\mathbb{N})$; that is, g is a function mapping natural numbers to sets of natural numbers. We will define g by the rule $g(n) = \{n, n + 1, n + 2, \ldots, 2n\}$, so for instance $g(0) = \{0\}$ and $g(3) = \{3, 4, 5, 6\}$. If we truly desired, we could list g directly, although doing so is perhaps less enlightening than the definition in terms of how it transformed each possible argument was: $g = \big\{(0, \{0\}), (1, \{1, 2\}), (2, \{2, 3, 4\}), (3, \{3, 4, 5, 6\}), \ldots\big\}$ Now, to identify the components of this function, definitionally, $\mathbb{N}$ is the domain and $\mathcal{P}(\mathbb{N})$ the codomain. But to find the image, we need to collect all the possible outputs of the function, e.g.: $\big\{\{0\}, \{1, 2\}, \{2, 3, 4\}, \{3, 4, 5, 6\}, \ldots\big\}$.

One more basic example: consider $h : \mathbb{R} \to \mathbb{R}$ given by the rule $h(x) = x^2$. As presented, h has domain $\mathbb{R}$ and codomain $\mathbb{R}$, but as is well known, the square of a real number is a *positive* real number, so the image of h is actually not $\mathbb{R}$ but $[0, \infty)$.

So much for simple examples. Now we get to the meat of function-description. There are two properties a function can have, describing where the elements of its domain are mapped, and answering two questions, specifically: (1) are two elements of the domain mapped to the same element of the codomain, and (2) is every element of the codomain mapped to?

Definition 1. A function $f : A \to B$ is described as *injective* or *one-to-one* if, for any $x, y \in A$ such that $x \neq y$, it is the case that $f(x) \neq f(y)$. An injective function may also be called an *injection*.

Definition 2. A function $f : A \to B$ is described as *surjective* or *onto* if the codomain and image of the function are identical. A surjective function may also be called an *surjection*.