1 Results on Injectivity and Surjectivity

To get a better handle on injectivity and surjectivity, we'll note several results about what these mean in a finite context:

Proposition 1. *If $f : A \to B$ is injective and B is a finite set, then A is a finite set and $|A| \leq |B|$.*

Proof. Let us define $n = |B|$ for simplicity. We shall proceed by contradiction: suppose that A is either infinite or a finite set with $|A| > |B| = n$; in both scenarios we may assert with confidence that A contains at least $n+1$ distinct elements, which we may refer to by the names $a_1, a_2, \ldots, a_{n+1}$. Since $a_1, a_2, \ldots, a_{n+1}$ are all distinct elements of A and f is an injection with domain A, we may be certain that $f(a_1), f(a_2), f(a_3), \ldots, f(a_{n+1})$ are all distinct, and since f has codomain B, we know that $f(a_1), f(a_2), f(a_3), \ldots, f(a_{n+1})$ are all elements of B. However, we have now identified $n+1$ distinct elements of B, which is an n-element set; such a thing is impossible. $\square$

so we may think of an injective function as requiring "room to spread out" into the codomain; if the codomain is too small, then a function could not be injective because inevitably multiple elements of the domain would have to be mapped to the same values.

It is important to realize that the converse of the proposition above is *not* true: $|A| \leq |B|$ is not sufficient to guarantee that $f : A \to B$ is injective! As an example of how this can be violated, consider $A = \{a, b\}$ and $B = \{x, y, z\}$. Certainly injective functions exist between these two sets, but so do many noninjective functions, such as $f = \{(a, y), (b, y)\}$.

We can prove a similar "relative-size" result for surjections

Proposition 2. *If $f : A \to B$ is surjective and A is a finite set, then B is a finite set and $|B| \leq |A|$.*

Proof. Let us define $n = |A|$ for simplicity. Let us denote the n elements of A by the names $a_1, a_2, \ldots, a_n$. The image of f is definitionally exactly the set consisting of the results of applying f to every element of $a_1, a_2, \ldots, a_n$, so the image of f is $\{f(a_1), f(a_2), f(a_3), \ldots, f(a_n)\}$. Note that the elements in this list may not be all distinct! Since f is surjective, its image must equal its codomain B, so $B = \{f(a_1), f(a_2), f(a_3), \ldots, f(a_n)\}$, which is clearly a finite set. Now $|B| = |\{f(a_1), f(a_2), f(a_3), \ldots, f(a_n)\}|$, which is the count of a list of n not necessarily distinct objects; the size of this list is thus no more than n, achieving equality only when all n items in the set are in fact distinct; thus $|B| \leq n = |A|$. $\square$

Of course, from these two results we can derive a straightforward and strong result on bijective functions:

Corollary 1. *If at least one of A or B is finite, and $f : A \to B$ is bijective, then both A and B are finite and $|A| = |B|$.*

Proof. If A is finite, then surjectivity of f guarantees B is finite; likewise, if B is finite, injectivity of f guarantees that A is finite, os if either A or B is finite, then both must be.

Since both sets are finite, we may make comprehensible statements referring to their sizes: by injectivity of f, $|A| \leq |B|$, and by surjectivity of f, $|B| \leq |A|$. Thus $|A| = |B|$. $\square$

An interesting question at this point, of course, is "what could we say along these lines about injectivity, surjectivity, and bijectivity when A and B are infinite?". We're not confronting that now, but we will return to this idea, I assure you!

2 Compositions