1 The Cantor-Schröder-Bernstein Theorem

Hang on to your seats, because today we have a major result, and one that's a bit difficult to prove.

Theorem 1 (Dedekind 1887, Cantor 1895, Schröder 1896, Bernstein 1897, König 1906)**.** *For sets A and B, if $f : A \to B$ is an injective function and $g : B \to A$ is an injective function, then there is a bijective function $h : A \to B$.*

It's worth noting, of course, that this is a trivial result when A and B are finite: then $|A| \leq |B|$ and $|B| \leq |A|$, so $|A| = |B|$ and one could just artificially match up each a_i to b_i. However, proving this for possibly infinite sets is a tricky and dangerous endeavor.

Proof sketch: We will start by explicitly constructing a relation h, and then show that it is a bijective function.

Let us call the images of f and g to be B' and A' respectively. Then, considering f as a function from A to B', it is surjective by construction of B' and injective by injectivity of f, hence $f : A \to B'$ is bijective, and likewise the codomain-restricted function $g : B \to A'$ is bijective. Thus we can define inverse functions $f^{-1} : B' \to A$ and $g^{-1} : A' \to B$.

Now, for any element a of A, we can define a straightforward infinite sequence by alternatingly applying f and g to it:

$$a, f(a), g(f(a)), f(g(f(a))), g(f(g(f(a)))), \ldots$$

and can furthermore attempt to extend this sequence backwards:

$$\ldots, g^{-1}(f^{-1}(g^{-1}(a))), f^{-1}(g^{-1}(a)), g^{-1}(a), a, f(a), g(f(a)), f(g(f(a))), g(f(g(f(a)))), \ldots$$

Note that our backwards sequence is on much shakier ground, existencewise, than the forwards step: in order for $g^{-1}(a)$ to exist, a must be in A', which it might not be; likewise, in order for $f^{-1}(g^{-1}(a))$ to exist, $g^{-1}(a)$ must be in B', which is not guaranteed; we thus have a reasonable expectation that this backwards progression will eventually be unable to be extended further. There turn out to be four cases we need to address: it is possible that this sequence reaches a place where it repeats, that it extends infinitely in both directions without repetitions, or that it terminates with some early element in $A - A'$ or $B - B'$. In each of these cases, we will describe the behavior of h. □