1 Comparative set size

Let's look at some questions: which is "larger", $\mathbb{N}$ or $\mathbb{Z}$? How about $\mathbb{Z}$ and $\mathbb{Q}$? How about $\mathbb{Q}$ and $\mathbb{R}$?

Prior to the twentieth century, there weren't well-defined analytical tools for describing "size" of infinite sets, so these questions were approached purely philosophically. Most answers boiled down to one of two possibilities: either one asserted that if $A \subsetneq B$, then A was "smaller" than B, or one asserted that all infinite sets were the same size (namely, "infinite").

The former viewpoint has the obvious flaw that sets where one is not a subset of the other wouldn't be comparable; the latter is self-consistent but misses some interesting subtleties. As it turns out, the most efective measure of the "size" of a set is not going to be consistent with either of these viewpoints.

2 Functions as comparison mechanisms

Let us consider a thought experiment: I might have a collection of stones which I think is the same as the number of people in the room. If there were only 5 people, it would be easy for me to show this just by counting the people, counting the stones, and showing that the nubmers are equal. But if there were, for instance, a hundred people, trying to count out both groups would be tedious and possibly even error-prone. A clever way for me to do this is to ask each person to take a stone, and then if everybody picks one up and there are none left over, then there are clearly the same number of people as stones.

From a mathematical perspective, what we are doing in the first place is ascertaining that $|A| = |B|$ by calculating both $|A|$ and $|B|$, and showing that they are the same number. In the second case, we are building a bijection between the sets A and B, and asserting that the existence of that bijection guarantees equivalency of the sizes of A and B.

Recall that we previously proved a result directly relevant to this fact:

Theorem 1. *If either A or B is a finite set, and $f : A \to B$ is a bijection, then both A and B are finite and $|A| = |B|$.*

That is to say, this theorem asserts that *finite* sets related by a bijection are of equal size. We can extrapolate this idea to *define* comparison of size on infinite sets.

Definition 1. For sets A and B, we say that A and B are *of equal cardinality*, denoted idiomatically as $|A| = |B|$, if there is a bijection $h : A \to B$.

Please note that $|A| = |B|$ above is a purely idiomatic phrase; previously, this had been an equality between numeric values, but here we are using it as a set expression, not a reference to equality of the numbesr $|A|$ and $|B|$, which may not exist in context.

As an example of how such an equivalency might work, we could assert, for instance, that $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ and $\{0, 1, 2, 3, 4, \ldots\}$ are of equal size, by crafting a simple bijection between them.

We can note, at this juncture, that "being the same size" is well-behaved — namely, it's an equivalence relation.

Proposition 1. *Equality-of-cardinality is an equivalence relation.*

Proof. For any set A, the identity map $e : A \to A$ given by $e(a) = a$ is clearly a bijection, so $|A| = |A|$; thus equality-of-cardinality is reflecive.

To demonstrate symmetry, let us consider sets A and B such that $|A| = |B|$; thus we are asserting the existence of a bijection $f : A \to B$. Since f is a bijection, it has an inverse function $f^{-1} : B \to A$, which is also a bijection, so $|B| = |A|$.

Finally, to demonstrate transitivity, let us consider sets A, B, and C such that $|A| = |B|$ and $|B| = |C|$. These premises can be interpreted as asserting the existence of bijections $f : A \to B$ and $g : B \to C$. Then, $g \circ f$ is a bijection from A to C, so $|A| = |C|$. $\square$