1. **(10 points)** *We observed in class that when A is a finite set, $|\mathcal{P}(A)| = 2^{|A|}$. Explain in your own words why this is true.*

   We might be best served by looking at something specific but not overspecific: noting that, for instance, when $A = \{1, 2, 3\}$, we can compute $\mathcal{P}(A)$ to have 8 elements is overspecific, because it really generalizes, at best, to talking about other 3-element sets. Let's fix some natural number n, and discuss $A = \{1, 2, 3, \ldots, n\}$: this is a pretty general set, and if we want to talk about other sets of size n, we could just replace the 1, 2, and so forth with whatever the actual elements of A are.

   So, with our specified but not over-specified choice of A, noting that $|A| = n$, we want to get down to the business of explaining why $|\mathcal{P}(A)|$ has 2^n elements. Or, more to the point, since the elements of $\mathcal{P}(A)$ are the subsets of A, our question becomes: why does A have 2^n different subsets?

   Let's consider the different ways that we can build a subset B of $\{1, 2, 3, \ldots, n\}$. B can only have the elements of A as elements, so we know it is uniquely determined by whether each of $1, 2, 3, \ldots, n$ is in B or not in B. For example, there are two possible relationships between 1 and B: either $1 \in B$ or $1 \notin B$. Likewise, there are two possible relationships between 2 and B: either $2 \in B$ or $2 \notin B$. And so forth, with each number from 1 to n being in one of two relationships with B. Thus, B is determined by n different choices between 2 alternatives. This can be done in $2 \times 2 \times 2 \times \cdots \times 2 = 2^n$ ways, so there are 2^n different ways to construct a subset B of A; thus A has 2^n different subsets.

2. **(5 points)** *Explain why, when $A \subseteq B$ and A and B are both finite sets, it follows that $|A| \leq |B|$.*

   If $A \subseteq B$, then every element of A is in B. Since every element of A is in B, then in the process of counting the elements of B, we would count every single element of A, so counting the elements in B would necessarily produce a number as large or higher than the number of elements of A.

3. **(4 points)** *Give examples of sets satisfying each of the conditions below:*

   (a) $S \subseteq \mathcal{P}(\mathbb{N})$.

   $\mathcal{P}(\mathbb{N})$ is a set which has all subsets of $\mathbb{N}$ as elements. A subset S of $\mathcal{P}(\mathbb{N})$ is thus a set which has (not necessarily all) subsets of $\mathbb{N}$ as elements. A simple example of such a set would be $S = \big\{\{3\}, \{1, 4\}, \{2\}\big\}$.

   Some more esoteric examples would be $S = \emptyset$ (which happens to contain zero subsets of $\mathbb{N}$), $S = \big\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \ldots\big\}$ (which contains an infinite number of two-element subsets of $\mathbb{N}$), or $S = \{\emptyset, \mathbb{N}\}$, which contains two subsets of $\mathbb{N}$: the empty set and $\mathbb{N}$ itself.

   (b) $T \in \mathcal{P}(\mathbb{N})$.

   By way of contrast to the above question, while a subset of $\mathcal{P}(\mathbb{N})$ was a set containing subsets of $\mathbb{N}$, an element of $\mathcal{P}(\mathbb{N})$ is just a single subset of $\mathbb{N}$. There are plenty of these, and a simple example is $T = \{2, 3, 5, 7\}$.

   Plenty of more exotic examples exist, such as $T = \emptyset$, $T = \{2, 4, 6, 8, 10, \ldots\}$, $T = \{1, 4, 9, 16, 25, 36, 49, \ldots\}$, and even $T = \mathbb{N}$.

   (c) $A \subseteq \mathcal{P}(\mathbb{N})$ *and* $|A| = 5$.

   This is very similar to part (a), only with the restriction that our chosen set must contain 5 elements. Since the elements are themselves subsets of $\mathbb{N}$, we need to just build a set containing five subsets of $\mathbb{N}$, such as $A = \{\{2,3\},\{1,5,7\},\{1\},\{6\},\emptyset\}$. You can make the individual subsets more exotic, if you prefer.

(d) *$B \in \mathcal{P}(\mathbb{N})$ and $|B| = 5$.*

This is very similar to part (b), only with the restriction that our chosen set must contain 5 elements. Since our set must be a subset of $\mathbb{N}$, we need to just build a five-element subset of $\mathbb{N}$, such as $B = \{1,3,6,10,15\}$.

4. **(6 points)** *Explain why, for any sets A and B, it must always be the case that $A \cap B \subseteq A \subseteq A \cup B$. Are there any situations where $A \cap B$ is not a proper subset of A, or where A is not a proper subset of $A \cup B$?*

There are two separate subset-relationships here: $A \cap B \subseteq A$, and $A \subseteq A \cup B$. We shall handle each of them separately.

First, we want to justify the assertion $A \cap B \subseteq A$. We shall do so by unrolling each part of it until we arrive at a vacuously true statement. Let us start by noting that, definitionally, $X \subseteq Y$ means "every element of X is an element of Y", so we can rephrase $A \cap B \subseteq A$ in words as "every element of $A \cap B$ is an element of A". We can now use another definition to expand this phrasing further: we know that to be an element of $A \cap B$, something must be an element of both A and B, so our phrase above can be further expanded into the statement "anything that is both an element of A and an element of B is an element of A" — a statement which is obviously true! We have thus demonstrated that $A \cap B \subseteq A$ is true, since written out in words it becomes a trivial statement.

We can demonstrate $A \subseteq A \cup B$ in much the same way. Again making use of the definition of the subset relation, we can rephrase $A \subseteq A \cup B$ in words as "every element of A is an element of $A \cup B$". We now use another definition to expand this phrasing further: to be an element of $A \cup B$, something must be an element of either A or B or both, so our phrase above can be further expanded into the statement "anything that is an element of A is either an element of A or B or both" — again a statement which is self-evident.

In answer to the final part of the question, we wonder when either of these subset-relationships can be nonproper. In order for $X \subseteq Y$ to *not* describe a proper subset, it must be the case that $X = Y$ (since if $X \neq Y$, the subset relationship is definitionally proper). So we are wondering when it is the case that $A \cap B = A$ or $A = A \cup B$.

In order for $A \cap B$ and A to be equal, their elements must be the same, so every element of A must be an element of both A and B; clearly every element of A is an element of A, so the real restriction here is that every element of A is an element of B, or, in other words, that A is a subset of B. Thus $A \cap B = A$ when $A \subseteq B$.

In order for A and $A \cup B$ to be equal, their elements must be the same, so everything that is an element of A or an element of B must be an element of A; clearly every element of A is an element of A, so the real restriction here is that every element of B is an element of A, or, in other words, that B is a subset of A. Thus $A = A \cup B$ when $B \subseteq A$.

5. **(6 points)** *For each real number r, define $A_r = \{r^2\}$, define B_r as the closed interval $[r-1, r+1]$, and define C_r as the open interval (r, ∞). For $S = \{1,2,4\}$, evaluate the following expressions:*

(a) *$\bigcup_{\alpha \in S} A_\alpha$ and $\bigcap_{\alpha \in S} A_\alpha$.*

(b) *$\bigcup_{\alpha \in S} B_\alpha$ and $\bigcap_{\alpha \in S} B_\alpha$.*

(c) $\bigcup_{\alpha\in S} C_\alpha$ *and* $\bigcap_{\alpha\in S} C_\alpha$.

Each of these will actually simplify to a cute finite intersection or union, since S is a finite set, and thus:

$$\bigcup_{\alpha\in S} A_\alpha = A_1 \cup A_2 \cup A_4 = \{1\} \cup \{4\} \cup \{16\} = \{1, 4, 16\}$$

$$\bigcap_{\alpha\in S} A_\alpha = A_1 \cap A_2 \cap A_4 = \{1\} \cap \{4\} \cap \{16\} = \emptyset$$

$$\bigcup_{\alpha\in S} B_\alpha = B_1 \cup B_2 \cup B_4 = [0, 2] \cup [1, 3] \cup [3, 5] = [0, 5]$$

$$\bigcap_{\alpha\in S} B_\alpha = B_1 \cap B_2 \cap B_4 = [0, 2] \cap [1, 3] \cap [3, 5] = \emptyset$$

$$\bigcup_{\alpha\in S} C_\alpha = C_1 \cup C_2 \cup C_4 = (1, \infty) \cup (2, \infty) \cup (4\infty) = (1, \infty)$$

$$\bigcap_{\alpha\in S} C_\alpha = C_1 \cap C_2 \cap C_4 = (1, \infty) \cap (2, \infty) \cap (4\infty) = (4, \infty)$$

6. **(4 points)** *Find an indexed collection of distinct sets* $\{A_n\}_{n\in\mathbb{N}}$ *(so that no two sets are equal) satisfying the following two conditions:*

$$\bigcap_{n=1}^{\infty} A_n = \{-1, 0, 1\} \text{ and } \bigcup_{n=1}^{\infty} A_n = \mathbb{Z}$$

We can translate these two conditions into two simple properties. In order for $\bigcap_{n=1}^{\infty} A_n$ to be $\{-1, 0, 1\}$, it must be the case that -1, 0, and 1 are elements of *each* A_n; in order for $\bigcup_{n=1}^{\infty} A_n$ to be $\mathbb{Z}$, every integer must appear in *at least one* A_n. There are several ways to do this, but the most obvious way to do it is to let $A_n = \{-n, -n+1, -n+2, \ldots, -1, 0, 1, \ldots, n-2, n-1, n\}$. Then it is obvious that any integer k is an element of $A_{|k|}$, and that -1, 0, and 1 are elements of every A_n for integers $n \geq 1$.

7. **(5 points)** *Give an example of a partition of* $\mathbf{Q}$ *into three subsets.*

There are a great many ways to do this. One absurd way would be to select two rational numbers to be in their own partitions, and put everything else in the third, for example:

$$\{\{0\}, \{1\}, \mathbb{Q} - \{0, 1\}\}$$

which is silly but is technically a partition. There is a more natural and less silly partition which makes use of a trichotomy of not just the rationals, but the reals and integers: each rational number must be exactly one of positive, negative, or zero, so there's an obvious partition into the three sets consisting of zero alone, the positive rationals, and the negative rationals:

$$\{\{0\}, \{x \in \mathbb{Q} : x > 0\}, \{x \in \mathbb{Q} : x < 0\}\}$$

8. **(5 point bonus)** *I briefly discussed self-reference as a problematic issue in class: here we can look at what makes it a problem. Let us consider, hypothetically, the concept of a set* A *containing all sets. Since* A *is itself a set, it would be the case that* $A \in A$ *(it would also be the case that* $\emptyset \in A$ *and* $\mathcal{P}(A) \in A$*, for those are both sets too).*

   So far this is not a problem. But now let us consider $S = \{X \in A : X \notin X\}$*. Clearly, for example,* $A \notin S$*, because as we saw above,* $A \in A$*. On the other hand, for instance,* $\emptyset$ *and* $\mathbb{Z}$ *would be in* S*, since neither the empty set nor the integers have themselves as members.*

   The key question: is it true or false that $S \in S$*, and what would investigating this question tell us?*

   The question of whether S is an element of itself is intrinsically unanswerable. S consists of those sets which are not elements of themselves, by its definition. Then, $S \in S$ only if $S \notin S$, and vice versa, which suggests that neither $S \in S$ nor $S \notin S$ can be true.

   This is a problem which haunted the early twientieth century under a number of guises, spawning theory and philosophy too wide-ranging to be gone into here. The immediate practical upshot of this investigation was to force mathematics to diminish its definition of what a "set" was considerably, to avoid such paradoxes as presented here.

   For further elaboration on this problem, do research on either of its common names in literature, where it is usually referred to as *the Barber Paradox* or *Russell's Paradox*.