This assignment is due on **Friday, January 21, 2011**.

1. **(10 points)** We observed in class that when A is a finite set, $|\mathcal{P}(A)| = 2^{|A|}$. Explain in your own words why this is true.

2. **(5 points)** Explain why, when $A \subseteq B$ and A and B are both finite sets, it follows that $|A| \leq |B|$.

3. **(4 points)** Give examples of sets satisfying each of the conditions below:

   (a) $S \subseteq \mathcal{P}(\mathbb{N})$.

   (b) $T \in \mathcal{P}(\mathbb{N})$.

   (c) $A \subseteq \mathcal{P}(\mathbb{N})$ and $|A| = 5$.

   (d) $B \in \mathcal{P}(\mathbb{N})$ and $|B| = 5$.

4. **(6 points)** Explain why, for any sets A and B, it must always be the case that $A \cap B \subseteq A \subseteq A \cup B$. Are there any situations where $A \cap B$ is not a proper subset of A, or where A is not a proper subset of $A \cup B$.

5. **(6 points)** For each real number r, define $A_r = \{r^2\}$, define B_r as the closed interval $[r-1, r+1]$, and define C_r as the open interval (r, ∞). For $S = \{1, 2, 4\}$, evaluate the following expressions:

   (a) $\bigcup_{\alpha \in S} A_\alpha$ and $\bigcap_{\alpha \in S} A_\alpha$.

   (b) $\bigcup_{\alpha \in S} B_\alpha$ and $\bigcap_{\alpha \in S} B_\alpha$.

   (c) $\bigcup_{\alpha \in S} C_\alpha$ and $\bigcap_{\alpha \in S} C_\alpha$.

6. **(4 points)** Find an indexed collection of distinct sets $\{A_n\}_{n \in \mathbb{N}}$ (so that no two sets are equal) satisfying the following two conditions:

$$\bigcap_{n=1}^{\infty} A_n = \{-1, 0, 1\} \text{ and } \bigcup_{n=1}^{\infty} A_n = \mathbb{Z}$$

7. **(5 points)** Give an example of a partition of $\mathbf{Q}$ into three subsets.

8. **(5 point bonus)** I briefly discussed self-reference as a problematic issue in class: here we can look at what makes it a problem. Let us consider, hypothetically, the concept of a set A containing all sets. Since A is itself a set, it would be the case that $A \in A$ (it would also be the case that $\emptyset \in A$ and $\mathcal{P}(A) \in A$, for those are both sets too).

   So far this is not a problem. But now let us consider $S = \{X \in A : X \notin X\}$. Clearly, for example, $A \notin S$, because as we saw above, $A \in A$. On the other hand, for instance, $\emptyset$ and $\mathbb{Z}$ would be in S, since neither the empty set nor the integers have themselves as members.

   The key question: is it true or false that $S \in S$, and what would investigating this question tell us?