1. **(10 points)** *Consider the statements* $P : x^2 - 3x + 2 = 0$ *and* $Q : x \geq 0$.

   (a) **(6 points)** *Explain in words why* $P \Rightarrow Q$ *is true.*

   If we were to presume that x satisfies the equation $x^2-3x+2 = 0$, then it follows from basic algebraic methods (the quadratic formula, factorization, or any other quadratic-solution method) that x is equal to either 1 or 2. In both of these cases, $x \geq 0$, so we have verified that whenever P is true, Q is true—which is exactly equivalent to asserting that $P \Rightarrow Q$ is true.

   (b) **(4 points)** *There are four pairs of truth values for* P *and* Q. *For each of the four pairs, either find a value of* x *corresponding to those truth values, or explain why such an* x *cannot exist.*

   A simple example where P is true and Q is true, as explored in the previous part of this problem, is $x = 1$ ($x = 2$ would also work). When $x = 1$, both $x^2 - 3x + 2 = 0$ and $x \geq 0$ are true statements.

   We can likewise find a simple example where P is false and Q is true by choosing any non-negative number other than 1 or 2. For instance, if we select $x = 3$, we can verify that $x^2 - 3x + 2 \neq 0$ (specifically, $x^2 - 3x + 2 = 2$), so P is false, but $x \geq 0$, so Q is true.

   Letting x be any negative number will result in false P and false Q. For instance, when $x = -1$, we see that $x^2 - 3x + 2 \neq 0$ (specifically, $x^2 - 3x + 2 = 6$), so P is false, and in addition, $x \ngeq 0$, so Q is also false.

   The one remaining case, where P is true and Q is false, is the truly interesting one. We *can't* find an example of a value of x such that $x^2-3x+2 = 0$ but $x \ngeq 0$, and this could be explained either algebraically, or, more relevantly, in terms of the assertion of the previous section. We saw that $P \Rightarrow Q$ was true, or in other words, whenever P is true, Q must be true. Thus, the pair of a true P and a false Q is not actually an achievable combination.

2. **(9 points)** *The statements* $(\neg P) \vee (\neg Q)$ *and* $\neg(P \wedge Q)$ *are equivalent. You will demonstrate this two ways.*

   (a) **(4 points)** *Fill in the following truth table, and note that the two columns corresponding to* $(\neg P) \vee (\neg Q)$ *and* $\neg(P \wedge Q)$ *have identical entries.*

P	Q	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

   We can observe visually that the fifth and seventh columns have identical truth values for all combinations of P and Q; thus, the expressions they describe are identical.

   (b) **(5 points)** *Write out the statements* $(\neg P) \vee (\neg Q)$ *and* $\neg(P \wedge Q)$ *in words instead of symbols, and explain why these two different statements describe the same situation.*

   $(\neg P) \vee (\neg Q)$ could be unrolled into "at least one of "not P" or "not Q" is true". Since "not P" is true only when P is false, and likewise for Q, we could rephrase this as "at least one of P or Q is false".

   Similarly, we can unroll $\neg(P \wedge Q)$ into "it is not the case that P and Q are both true". Showing these two expressions are equivalent is, after a fashion, common sense: P and Q will not both be true exactly when at least one of them is not true.

3. **(6 points)** *Letting P and Q be statements, identify each of the following statements as either tautological, contradictory, or neither. Justify your results, either with explanation or exhaustive computation.*

- $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$.

  This statement could be read as "if both P and $P \Rightarrow Q$ are true, then Q is true". This will actually be a tautology, because $P \Rightarrow Q$ expands to mean "when P is true, Q is also true", so the full statement is "if P is true and in addition, when P is true, Q is true, it follows that Q is true", something which is self-evident.

  Alternatively, one could compute the truth table for this statement and note that it is true regardless of P and Q.

P	Q	$P \to Q$	$P \wedge (P \Rightarrow Q)$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

  As a point of useful information, this statement, which is called *modus ponens* (Latin: the method of affirmation), is one of the building blocks of proof methodology. Most important results are presented in the form "if P than Q". Often, we want to use such a theorem, together with the knowledge that P is true, to assert Q. We shall see the use of this method later in the course.

- $(P \vee Q) \Leftrightarrow (P \wedge Q)$.

  This is neither a tautology nor a contradiction. It claims that "P or Q" has the same truth value as "P and Q", which is sometimes true (for instance, when P and Q are both true), and sometimes false (for instance, when P is true and Q is false).

- $(P \wedge Q) \Rightarrow \neg Q$.

  This is neither a tautology nor a contradiction. It seems moderately contradictory when written out, as it claims that if both P and Q are true, then Q is false, which seems like a statement that would always be false, but in fact, it is only given the lie when P and Q are both true, in which case the implication's premise is met and its consequence fails, so the implication is false. For *all* other values of P and Q, no contradiction arises since the premise isn't even true, so the implication is true regardless of the truth value of the consequence.

P	Q	$P \to Q$	$P \wedge (P \Rightarrow Q)$	$\neg Q$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
T	T	T	T	F	F
T	F	F	F	T	T
F	T	T	F	F	T
F	F	T	F	T	T

4. **(5 points)** *Let P be the proposition "n is a prime number between 4 and 10". For each of the propositions given below, indicate whether that proposition is a necessary condition for P, a sufficient condition for P, both, or neither; briefly justify your claim.*

- Q_1: *n is equal to 5.*

If Q_1 is satisfied, then $n = 5$, and thus P is satisfied, since 5 is indeed a prime number between 4 and 10. Thus $Q_1 \Rightarrow P$, so Q_1 is a sufficient condition for P.

However, it is not the case that if P is satisfied, then Q_1 must be satisfied: for instance, $n = 7$ satisfies Q_1 but not P. Thus it is not true that $P \Rightarrow Q_1$, so Q_1 is not a necessary condition for P.

- *Q_2: n is a positive, odd integer.*

  Even if Q_2 is satisfied, P might not be satisfied; for instance, $n = 9$ satisfies Q_2 but not P. Thus it is not the case that $Q_2 \Rightarrow P$, so Q_2 is not a sufficient condition for P.

  However, if P is satisfied, then Q_2 must be: we know every prime number is positive, and every prime number except 2 is odd, so any number satisfying the conditions of P will be a positive odd integer, satisfying Q_2. Thus $P \Rightarrow Q_2$, so Q_2 is a necessary condition for P.

- *Q_3: n is a prime number less than 6.*

  Even if Q_3 is satisfied, P might not be satisfied; for instance, $n = 2$ satisfies Q_3 but not P. Thus it is not the case that $Q_3 \Rightarrow P$, so Q_3 is not a sufficient condition for P.

  Likewise, even if P is satisfied, Q_3 might be unsatisfied, as by for instance $n = 7$, which makes P true but not Q_3. Thus it is not the case that $P \Rightarrow Q_3$, so Q_3 is not a necessary condition for P either.

5. **(5 points)** *Below, we shall discuss the true statement "For every rational number r, $\frac{1}{r}$ is rational."*

   (a) **(2 points)** *Write this statement entirely in symbols. Note that we can assert that some x is rational with the statement $x \in \mathbb{Q}$.*

   "For every rational number r" is a universal quantification, so it would be written as $\forall r \in \mathbb{Q}$. The statement "$\frac{1}{r}$ is rational" can be written symbolically as $\frac{1}{r} \in \mathbb{Q}$, so putting these two parts together, the above statement is:

   $$\forall r \in \mathbb{Q}, \frac{1}{r} \in \mathbb{Q}$$

   (b) **(3 points)** *Write the negation of this statement in words, in as easy-to-comprehend a way as possible. Do not simply wrap the entire expression in the phrase "it is not the case that...". Note that the statement you produce will in fact be false.*

   The negation of a universally quantified statement is an existentially quantified statement: that is, "for all x, P is true" has negation "for some x, P is false". In this particular case, the negation of the statement given is "For some rational number r, $\frac{1}{r}$ is not rational". Note that this statement is false: there in fact is no rational number whose reciprocal is not rational.

6. **(5 points)** *Below, we shall discuss the false statement "There is a rational number r such that $r^2 = 2$."*

   (a) **(2 points)** *Write this statement entirely in symbols.*

   "There is a rational number r" is an existential quantification, so it would be written as $\exists r \in \mathbb{Q}$. The statement "$r^2 = 2$" is already symbolic, so putting these two parts together, the above statement is:

   $$\exists r \in \mathbb{Q} : r^2 = 2$$

(b) **(3 points)** *Write the negation of this statement in words, in as easy-to-comprehend a way as possible. Do not simply wrap the entire expression in the phrase "it is not the case that...". Note that the statement you produce should be true.*

The negation of an existentially quantified statement is a universally quantified statement: that is, "for some x, P is true" has negation "for all x, P is false". In this particular case, the negation of the statement given is "For all rational numbers r, $r^2 \neq 2$". This is a true statement, which you probably already know because $\pm\sqrt{2}$ are well-known to be irrational; however, we will actually prove it later in the course.

7. **(4 point bonus)** *Logic is important in designing computers, since the "true" and "false" properties of a statement correspond to the circuit states of having a high signal (usually 5 volts) and a low signal (0 volts). Due to technical restrictions, early computers had only one basic operation, generally called NAND (standing for "not-and") and written with the symbol* $\uparrow$. $P \uparrow Q$ *was logically equivalent to* $\neg(P \wedge Q)$, *as shown in the following truth table.*

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

While this was the only primitive operation which was feasible, early computer scientists of course wanted to use the more familiar and useful operations. Show that one can express the statements $\neg P$, $P \wedge Q$, *and* $P \vee Q$ *entirely in terms of repeated application of the "nand" operation to* P *and* Q *in various combinations.*

It's actually fairly easy to construct $\neg P$: since $P \wedge P = P$, we know that

$$\neg P = \neg(P \wedge P) = P \uparrow P$$

And now that we have negation, it's not too hard to bootstrap $P \wedge Q$ by expanding it into a collection of nands and negations:

$$P \wedge Q = \neg\neg(P \wedge Q) = \neg(P \uparrow Q) = (P \uparrow Q) \uparrow (P \uparrow Q)$$

The trickiest statement to construct is $P \vee Q$. We can appeal to DeMorgan's Law: $P \vee Q = \neg[(\neg P) \wedge (\neg Q)]$, and since we already have both the negation and conjunction operations expressable in terms of the nand operation, we can simply unroll this piece by piece:

$$P \vee Q = \neg[(\neg P) \wedge (\neg Q)] = (\neg P) \uparrow (\neg Q) = (P \uparrow P) \uparrow (Q \uparrow Q)$$

"I know what you're thinking about," said Tweedledum: "but it isn't so, nohow." "Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."

—Lewis Carroll, *Through the Looking-Glass*