

1. **(12 points)** Below are three existence statements which are either true or false. For each of them, either prove them true (by either an example or a nonconstructive proof) or prove them false (by a disproof of existence).

- (a) There is a real number x such that $x^6 + x^4 + 1 = 2x^2$.

This statement is false. The disproof follows.

Proof. We may rephrase the above equation as $x^6 + x^4 - 2x^2 + 1 = 0$; the last three terms invite a factorization into the form $x^6 + (x^2 - 1)^2$. We know that the square x^6 is greater than or equal to 0 for all real x , achieving equality only when $x = 0$; likewise the square $(x^2 - 1)^2$ is greater than or equal to zero for all real x , achieving equality only when $x^2 - 1 = 0$ (i.e. at the values $x = -1$ and $x = 1$). Thus, for all real x except 0, 1, and -1 , it is clear that $x^6 > 0$ and $(x^2 - 1)^2 > 0$; thus for all real x except 0, 1, and -1 , $x^6 + (x^2 - 1)^2 > 0$. Also, we may specifically compute that $0^6 + (0^2 - 1)^2 > 0$, $1^6 + (1^2 - 1)^2 > 0$, and $(-1)^6 + ((-1)^2 - 1)^2 > 0$, so for all real x , $x^6 + (x^2 - 1)^2 > 0$, so there is no real solution to the equation $x^6 + x^4 - 2x^2 + 1 = 0$. \square

- (b) There is an integer n such that $4 \mid n^2 + 2$.

This statement is false. The disproof follows.

Proof. The most effective approach is casewise on the parity of n :

Case I: n is even. Then $n = 2k$ for some integer k , and thus $n^2 + 2 = (2k)^2 + 2 = 4k^2 + 2$. By the definition of divisibility $4 \mid 4k^2$, but $4 \nmid 2$, so $4 \nmid 4k^2 + 2$.

Case II: n is odd. Then $n = 2k + 1$ for some integer k , and thus $n^2 + 2 = (2k + 1)^2 + 2 = 4k^2 + 4k + 3$. By the definition of divisibility $4 \mid 4k^2 + 4k$, but $4 \nmid 3$, so $4 \nmid 4k^2 + 4k + 3$. \square

- (c) There is an integer n such that $n^3 \equiv 6 \pmod{7}$.

This statement is true, and we demonstrate the existence of such an n with the specific value $n = 3$. $3^3 = 27$, and since $7 \mid (27 - 6)$, it follows by definition that $3^3 \equiv 6 \pmod{7}$. Other specific values may also work.

2. **(6 points)** Prove that there is exactly one solution to the equation $x = \cos x$.

Proof. The solutions to the given equation are identically the solutions to $x - \cos x = 0$, so if we let $f(x) = x - \cos x$, the above assertion is identical to the assertion that $f(x)$ has exactly one zero.

We prove that $f(x)$ has at least one zero with the Intermediate Value Theorem. $f(0) = 0 - \cos 0 = -1$, while $f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos \frac{\pi}{2} = \frac{\pi}{2}$. Since $f(0)$ is negative and $f(\frac{\pi}{2})$ is positive, and $f(x)$ is a continuous function, it follows from the Intermediate Value Theorem that $f(c) = 0$ for some value of $c \in [0, \frac{\pi}{2}]$.

To prove that this zero is unique, we may appeal to the fact that $f(x)$ is increasing throughout, which is easily proven with calculus: $f'(x) = 1 - \sin x$. Since $\sin x \leq 1$, $f'(x) \geq 1 - 1 = 0$, so $f(x)$ is increasing. Thus, if we had two zeroes a and b of $f(x)$ with $a < b$, the criterion that $f(x)$ is increasing would imply that $f(a) < f(b)$; however, since they are both zeroes, this would be equivalent to the absurdity $0 < 0$. \square

3. **(6 points)** Prove that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for every positive integer n .

Proof. We shall prove the above statement by induction on n . The base case $n = 1$ is easily verified arithmetically: $1 \leq 2 - \frac{1}{1}$.

Let us fix a value k , and assume by the inductive hypothesis that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$. Since we want to place a bound on $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2}$, we shall add $\frac{1}{(k+1)^2}$ to both sides of this inequality, and proceed as necessary:

$$\begin{aligned} 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\ 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &\leq 2 + \frac{k - (k+1)^2}{k(k+1)^2} \\ 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &\leq 2 - \frac{k^2 + k + 1}{k(k+1)^2} \\ 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &< 2 - \frac{k^2 + k}{k(k+1)^2} \\ 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &< 2 - \frac{1}{k+1} \end{aligned}$$

Note that in the second to last step above we “nudged” our inequality, increasing the right side by $\frac{1}{k(k+1)^2}$, which is permissible: if something is bounded above by a certain quantity, it is also bounded above by a slight increase in that quantity. We have thus completed our inductive step. □

A note for the idly curious: better upper bounds than those approaching 2 are in fact possible for the infinite series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$. It was originally proven by Euler in 1741 using a clever power series decomposition that this infinite sum is actually $\frac{\pi^2}{6}$; a number of proofs of this result using other methods have been discovered since.

4. (8 points) Consider the sequence given by the recurrence:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 4 \\ a_3 &= 9 \\ a_n &= a_{n-1} - a_{n-2} + a_{n-3} + 4n - 6 \text{ for } n \geq 4 \end{aligned}$$

Explore the next few values of the recurrence and conjecture a formula to explain their values. Then, prove your conjecture.

We calculate a few values:

$$\begin{aligned} a_4 &= 9 - 4 + 1 + 16 - 6 = 16 \\ a_5 &= 16 - 9 + 4 + 20 - 6 = 25 \end{aligned}$$

which confirms the suspicion which we probably already had, that $a_n = n^2$. We shall prove this below.

Proof. We shall prove that $a_n = n^2$ for all integer $n \geq 1$ by induction. Our base cases from $n = 1$ to $n = 3$ are simply stated in our recurrence: $a_1 = 1 = 1^2$, $a_2 = 4 = 2^2$, and $a_3 = 9 = 3^2$. For the inductive step we fix a value $k \geq 3$ and assume that for all positive integers $n \leq k$, $a_n = n^2$; specifically we shall make use of the assumption that $a_k = k^2$, $a_{k-1} = (k-1)^2$, and $a_{k-2} = (k-2)^2$. Armed with these facts, we attempt to calculate a_{k+1} by invoking its definition according to the recurrence:

$$\begin{aligned} a_{k+1} &= a_{(k+1)-1} - a_{(k+1)-2} + a_{(k+1)-3} + 4(k+1) - 6 \\ &= a_k - a_{k-1} + a_{k-2} + 4k + 4 - 6 \\ &= k^2 - (k-1)^2 + (k-2)^2 + 4k - 2 \\ &= k^2 - (k^2 - 2k + 1) + (k^2 - 4k + 4) + 4k - 2 \\ &= k^2 + 2k + 1 = (k+1)^2 \end{aligned}$$

completing our inductive step. □

5. (8 points) *The Fibonacci numbers are given by the recurrence:*

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 3 \end{aligned}$$

Prove that F_n is even if and only if n is divisible by 3.

There is a straightforward fact which nonetheless it behooves us to prove separately:

Lemma 1. *If n is a positive integer, then exactly one of the three numbers n , $n-1$, or $n-2$ is divisible by 3.*

Proof. There are several ways to prove this; in deference to the theme of the week, we shall prove it by induction on n . The base case $n = 1$ is trivially true: of the numbers 1, 0, and -1 , exactly one (specifically, 0) is divisible by 3.

For the inductive step, let us fix a positive integer k , and take as the inductive hypothesis the presumption that exactly one of k , $k-1$, or $k-2$ is divisible by 3. It is clear that $3 \mid (k-2) + 3$ if and only if $3 \mid (k-2)$, since $3 \mid \pm 3$ and the sum of two multiples of 3 will be a multiple of 3. Thus, since exactly one of k , $k-1$, or $k-2$ is divisible by 3, it also follows that exactly one of k , $k-1$, or $(k-2) + 3$ is divisible by 3. These three quantities can be reordered and rephrased to be $(k+1)$, $(k+1) - 1$, and $(k+1) - 2$, so we have completed our inductive step. □

Now we may prove our main result:

Proposition 1. *For every positive integer n , F_n is even if and only if n is divisible by 3.*

Proof. We will prove this result by induction on n . For the base cases $n = 1$ and $n = 2$, this result is trivially true: $3 \nmid 1$, and F_1 is not even; likewise, $3 \nmid 2$, and F_2 is not even.

Now, let us fix a value of k , and assume that it is already known for all positive $n \leq k$ that F_n is even if and only if n is divisible by 3. We now want to show that the same parity criterion

applies to F_{k+1} . There are two possibilities we should confront with regard to the possible value of $k + 1$, which we specify casewise:

Case I: $k + 1$ is divisible by 3. Then, by Lemma 1, neither k nor $k - 1$ is divisible by 3, so by our inductive hypothesis, F_k and F_{k-1} will be both odd. Thus, $F_{k+1} = F_k + F_{k-1}$ is the sum of two odd numbers, and is thus even.

Case II: $k + 1$ is not divisible by 3. Then, by Lemma 1, exactly one of $k + 1$, k and $k - 1$ is divisible by 3. Clearly $k + 1$ is not divisible by 3 by our case hypothesis, so exactly one of k or $k - 1$ is divisible by 3. Thus by our inductive hypothesis, exactly one of F_k and F_{k-1} will be even (and the other will be odd). Thus, $F_{k+1} = F_k + F_{k-1}$ is the sum of an odd and even number, and is thus odd.

We have thus proven our inductive step. □

6. **(4 point bonus)** *The idyllic village of Salemo (population 150) has inhabitants who are happy, healthy, and expert logicians. They believe (oddly enough) that their prosperity and intelligence derives from a specific sort of ignorance: none of the villagers knows their own eye color. As none of the inhabitants wishes to ruin the others' success, anyone who does find out their own eye color during the day will leave town quietly at night (although everyone will learn that they left by the following morning). To avoid such a misfortune, they never speak of eye color and have no mirrors. However, one fine day, a stranger indiscreetly made reference to a blue-eyed villager (but not by name, or in any other way which identified a specific person). As it so happens, 50 of the villagers have blue eyes and the remaining 100 have brown eyes, and every villager knows the eye color of every other villager. What, if anything, will result from the stranger's indiscretion?*

We might investigate, by way of getting a handle on the problem, what happens in a village with only one blue-eyed person after such an utterance. Such a person, looking around them and seeing only non-blue eyes, would immediately conclude that they are themselves the blue-eyed person, so they would be compelled to leave on the first night.

If we had two blue-eyed people, each would look around and see exactly one blue-eyed person, and each would reason as such: "either I am blue-eyed or that other person is the only blue-eyed person in the village; as reasoned above, if there's only one blue-eyed person, then they will leave on the first night, but I don't know whether I am blue-eyed or not, so I will not leave the first night." During the second day, each will notice that the other has not left, and thus will conclude that the other is *not* the only blue-eyed person, and that they themselves will need to be blue-eyed to explain this occurrence. Having figured out their own eye-color, they will both leave on the second night.

In fact, one can extend this same argument inductively to prove that all the blue-eyed people will leave on the n th night after the revelation, where n is the number of blue-eyed people.

Proposition 2. *In a village with in the same situation as Salemo with a positive number n of blue-eyed people, all the blue-eyed people will leave on the n th night.*

Proof. We proceed by induction on n . The base case was reasoned above: a lone blue-eyed individual will realize their eye-color immediately after the revelation, and leave the first night.

We now assume the inductive hypothesis that a village with some fixed number k of blue-eyed people will have an exodus on the k th night; furthermore, since the presumption is that we

arrived at this conclusion by logical means, we may assume the villagers *also* know this fact about a supposed village with k blue-eyed people.

A blue-eyed villager in a village with $k + 1$ blue-eyed people will see k other blue-eyed people, and will conclude: “either I am blue-eyed or there are exactly k blue-eyed people, none of whom are me. In the latter case, I can expect all of them to leave on the k th night (by the inductive hypothesis); however, I myself don’t know yet which of these two cases are true, so I won’t leave yet.” This logic will not induce any villagers to leave during the first k nights; on the $(k + 1)$ th day, when they observe that nobody has actually left, each blue-eyed villager will be forced to revise their logic and reject the case where there are exactly k blue-eyed people and instead embrace the view that there are exactly $k + 1$ blue-eyed people, of which they are one. Since each blue-eyed villager learns their own eye-color by this logic, they will all leave on the $(k + 1)$ th night. \square

So in our specific problem, the 50 blue-eyed people will leave on the fiftieth day. The brown-eyed people will not be able to reach any conclusions, since none of them are aware that blue and brown are the only two eye-colors, and each might be able to forever entertain the prospect that, for instance, their eyes are green and it is only the other 99 people whose eyes are brown.

The above result is in some ways paradoxical on inspection. The revelation, one might argue, contains no new information: every villager already knows that there are blue-eyed people in town. The information actually revealed (explaining the paradox) is in some ways subtler.

In the one-blue-eyed-person case, the information that there is at least one blue-eyed person is authentically news to one person, so there the information conveyed is direct. In the two-person case the situation is much more complicated: each person surely already knows that there is one blue-eyed person (since they can see them), but they do not know that that blue-eyed person can themselves see a blue-eyed person. In this case, the revealed information is not just “there is a blue-eyed person”, but the subtler and stronger bit of knowledge “everyone knows that there is a blue-eyed person”. With three blue-eyed people the relevant knowledge is subtler still: “everyone knows that everyone knows that there is a blue-eyed person”. The public nature of the revelation is such that the information revealed in our original problem incorporates not just the flat fact of eye color, but the knowledge, in each villager’s understanding, that every other villager is working from the same premise.

So, naturalists observe, a flea
 Has smaller fleas that on him prey,
 And these have smaller still to bite 'em,
 And so proceed *ad infinitum*.

—Jonathan Swift, “On Poetry: A Rhapsody”