1. **(12 points)** Below are three existence statements which are either true or false. For each of them, either prove them true (by either an example or a nonconstructive proof) or prove them false (by a disproof of existence).

   (a) There is a real number $x$ such that $x^6 + x^4 + 1 = 2x^2$.

   (b) There is an integer $n$ such that $4 \mid n^2 + 2$.

   (c) There is an integer $n$ such that $n^3 \equiv 6 \pmod 7$.

2. **(6 points)** Prove that there is exactly one solution to the equation $x = \cos x$.

3. **(6 points)** Prove that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for every positive integer $n$.

4. **(8 points)** Consider the sequence given by the recurrence:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 4 \\ a_3 &= 9 \\ a_n &= a_{n-1} - a_{n-2} + a_{n-3} + 4n - 6 \text{ for } n \geq 4 \end{aligned}$$

   Explore the next few values of the recurrence and conjecture a formula to explain their values. Then, prove your conjecture.

5. **(8 points)** The Fibonacci numbers are given by the recurrence:

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 3 \end{aligned}$$

   Prove that $F_n$ is even if and only if $n$ is divisible by 3.

6. **(4 point bonus)** The idyllic village of Salemo (population 150) has inhabitants who are happy, healthy, and expert logicians. They believe (oddly enough) that their prosperity and intelligence derives from a specific sort of ignorance: none of the villagers knows their own eye color. As none of the inhabitants wishes to ruin the others' success, anyone who does find out their own eye color during the day will leave town quietly at night (although everyone will learn that they left by the following morning). To avoid such a misfortune, they never speak of eye color and have no mirrors. However, one fine day, a stranger indiscreetly made reference to a blue-eyed villager (but not by name, or in any other way which identified a specific person). As it so happens, 50 of the villagers have blue eyes and the remaining 100 have brown eyes, and every villager knows the eye color of every other villager. What, if anything, will result from the stranger's indiscretion?

> So, naturalists observe, a flea
> Has smaller fleas that on him prey,
> And these have smaller still to bite 'em,
> And so proceed *ad infinitum.*
>
> —Jonathan Swift, "On Poetry: A Rhapsody"