1. **(6 points)** In power-of-3 nim, the game state is a single non-negative integer (i.e., a stack of coins), such that each move consists of subtracting some power of 3 (i.e., removing a number of objects which is a power of 3). For instance, in a game with initial state 200, the first player could reduce the state to 199, 197, 191, 173, or 119. The game is won by whoever reduces the state to zero (i.e. removes the last coin).

   (a) **(3 points)** Show that with an initial state of 200 and rational play by the second player, the first player will lose.

   (b) **(3 points)** Show that with an initial state of 200, the first player will lose regardless of how the second player plays.

2. **(10 points)** 2-or-3 nim is played with again a game state consisting of a single integer (i.e. a stack of coins), and moves consisting of subtracting 2 or 3 from the game state (i.e. removing 2 or 3 coins). The game is won by whichever player reduces the stack to either 1 or 0 coins (i.e. when their opponent cannot move). Experiment with this game to determine which states are winning and losing; then form and prove a conjecture about which states are winning and which are losing.

3. **(13 points)** We shall inspect the false statement: "For a set A and function $f : A \to A$, the function f is inejctive if and only if it is bijective.

   (a) **(3 points)** Demonstrate a set A and function $f : A \to A$ which is injective but not surjective.

   (b) **(4 points)** Demonstrate a set A and function $f : A \to A$ which is surjective but not injective.

   (c) **(6 points)** Find as unrestrictive a condition on A as possible to make the statement "A function $f : A \to A$ is injective if and only if it is surjective" true. Prove your result.

4. **(11 points)** The following questions relate to function composition.

   (a) **(6 points)** Prove or disprove: for any sets A, B, and C, with functions $f : A \to B$ and $g : B \to C$, if f is bijective and $g \circ f$ is surjective, then g is surjective.

   (b) **(5 points)** Prove or disprove: for any sets A, B, and C, with functions $f : A \to B$ and $g : B \to C$, if g is injective and $g \circ f$ is injective, then f is injective.

5. **(4 point bonus)** We craft a collection of sequences by the following procedure: we let $A_1 = (1)$, and then for each A_{i+1} henceforth, we read A_i left-to-right, counting the number of occurrences of each number, and then writing the numbers spoken as our new sequence. So, we would read A_1 as "one 1", and write out $A_2 = (1, 1)$. Then we read out A_2 as "two 1s", and write out $A_3 = (2, 1)$; reading out A_3 as "one 2, one 1" gives us $A_4 = (1, 2, 1, 1)$.

   Prove that the number 4 does not appear in any sequence A_i.

Problems worthy of attack prove their worth by hitting back. —Piet Hein, "Problems"