1. **(24 points)** *Demonstrate the existence of bijections between the following pairs of sets.*

   (a) **(4 points)** *The set $\mathbb{Z}$ and the set of positive even integers $E = \{2, 4, 6, 8, 10, \ldots\}$.*

   Let $f : \mathbb{Z} \to E$ be given by $f(x) = \begin{cases} 4x \text{ if } x > 0 \\ 2 - 4x \text{ if } x \leq 0 \end{cases}$.

   It is quite easy to see that this function is injective: if x and y were distinct positive integers, clearly $4x \neq 4y$; likewise if they were distinct nonpositive integers, then $2 - 4x \neq 2 - 4y$. Finally, if x is positive and y is nonpositive, then $f(x)$ would be divisible by 4 whereas $f(y)$ would not be. Thus, if $x \neq y$, then $f(x) \neq f(y)$.

   To see that it is surjective, let us consider an element y of E: if $4 \mid y$, then $y = 4k$ for some positive integer k and then $y = f(k)$. If $4 \nmid y$, then since $2 \mid y$, $y = 4k + 2$ for some positive integer k, so $y = f(-k)$.

   (b) **(5 points)** *The set $\mathbb{N}$ and the set A of quadratic functions with integer coefficients $\{ax^2 + bx + c : a, b, c \in \mathbb{Z}\}$*

   Let us divide the quadratics into classes based on the sum of the absolute value of their coefficients: let class $A_n = \{ax^2 + bx + c : a, b, c \in \mathbb{Z}, |a| + |b| + |c| = n\}$. Significantly, each class is finite: A_0 contains only 1 polynomial; A_1 contains 6 polynomials, A_2 contains 18, A_3 contains 38, and so forth (we could conservatively note that since every coefficient in A_n is between $-n$ and n, that $|A_n| \leq (2n + 1)^3$; this is a sloppy bound but enough to assure us each A_n is finite. We may then simply order each A_n (any way we like) and simply list them end-to-end to have an enumeration of all quadratics; that is, we let $f(1) = 0x^2 + 0x + 0$, let $f(2)$ through $f(7)$ be the elements of A_1 in any order we like, let $f(8)$ through $f(25)$ be the elements of A_2 in any order we like, and so forth. This is clearly both an injection and a surjection, since each quadratic appears in exactly one A_n, and thus will be listed exactly once.

   (c) **(5 points)** *The set $\mathbb{N}$ and the set S of all* finite *subsets of $\mathbb{N}$.*

   We can use the same trick as in the previous part of dividing our desired codomain into a countable number of finite parts which we then list end-to-end: let S_n contain those finite subsets A of $\mathbb{N}$ for which $\sum_{i \in A} i = n$; so, for instance, $\{1, 3, 4\} \in S_8$ since $1 + 3 + 4 = 8$. Significantly, each S_n is finite: S_0 contains only the empty set, S_1 contains only $\{1\}$, S_2 contains only $\{2\}$, S_3 contains $\{3\}$ and $\{1, 2\}$, and so forth (conservatively, it's easy to show $|S_n| \leq 2^n$; as above, this is massively inaccurate but sufficient to show that each S_n is finite). As above, given this partition of S into a countable number of finite parts, a bijection $\mathbb{N} \to S$ is easy to craft: we just list S_0, then S_1, then S_2, and so forth, and thus each natural number is associated with a distinct element of S, and since each S_i is finite, any given element of S will eventually be reached in this listing.

   An imaginative alternative solution involves associating a finite subset $\{a_1, a_2, \ldots, a_n\}$ of $\mathbb{N}$ with the natural number $2^{a_1} + 2^{a_2} + 2^{a_3} + \cdots + 2^{a_n}$. Since binary representations of natural numbers are uniquely associated with the numbers they represent, this is clearly a bijection; for instance, since 53 is uniquely representable in binary as $32 + 16 + 4 + 1 = 2^5 + 2^4 + 2^2 + 2^0$, it would be uniquely asssociated with the set $\{5, 4, 2, 0\}$.

   (d) **(5 points)** *The set $\mathbb{R}$ and the closed interval $[0, 1]$.*

   There are several possible ways to build a bijection here, most of which are best achieved using two injections and invoking the Cantor-Schröder-Bernstein Theorem. For instance, we could invoke $f(x) = \frac{1}{\pi} \arctan x + \frac{1}{2}$ as an easy injection from $\mathbb{R}$ to $[0, 1]$; we know from precalculus that $\arctan x$ maps the domain $\mathbb{R}$ injectively into $(\frac{\pi}{2}, \frac{\pi}{2})$; the transformations included in the statement of $f(x)$ guarantee that it has an image of $(0, 1)$ instead. Since we bijectively map $\mathbb{R}$ into the open interval $(0, 1)$, it is easy to recontextualize this as an injective map into $[0, 1]$ by simply adding 0 and 1 to the codomain (note that this will break surjectivity).

Now we craft the trivial injective map $g(x) = x$ from $[0, 1]$ to $\mathbb{R}$; at this point, since we have injective $f : \mathbb{R} \to [0, 1]$ and injective $g : [0, 1] \to \mathbb{R}$, the Cantor-Schröder-Bernstein Theorem guarantees existence of a bijection $h : \mathbb{R} \to [0, 1]$.

(e) **(5 points)** *The half-open interval* $[0, 1)$ *and the set* $[0, 1) \times [0, 1)$.

It is possible to build an explicit bijeciton here, but far easier to craft two injections and invoke Cantor-Schröder-Bernstein. We shall describe a real number $0 \leq x < 1$ as having a *standard decimal representation* $0.x_1x_2x_3x_4\ldots$ if there is no value i such that for all $j > i$, $x_j = 9$ (we define this standard to specifically ensure that, for instance, $\frac{3}{20}$ has a unique representation under consideration, since both $0.15000\ldots$ and $0.14999\ldots$ are decimal expansions describing this specific number.

Based on this concept of standard representation, we can craft a function $f : [0, 1) \times [0, 1) \to [0, 1)$ via the rule $f((x, y)) = 0.x_1y_1x_2y_2x_3y_3\ldots$. Clearly distinct values of x and y lead to distinct standard decimal representaitons, which produce distinct decimal sequences in $f((x, y))$; furthermore, since neither x nor y ends in an infinite sequence of 9s, neither will their interleaving, so we may be assured distinct decimal representations in $f((x, y))$ describe distinct real numbers; thus, given $(x, y) \neq (x', y')$, we may be assured that $f((x, y)) \neq f((x', y'))$, so the function f is injective. Note that due to our privileging of standard representations (which was necessary to make this function well-defined and certain to be injective), this function is *not* surjective: there is no pair (x, y) yielding the interlace decimal $0.1929392919090909\ldots$, for instance.

An injective function from $[0, 1)$ to $[0, 1) \times [0, 1)$, however, is easy to craft. Let $g(x)$ be defined as equal yo the ordered pair $(0, x)$; this is trivially injective, so by the Cantor-Schröder-Bernstein Theorem, the existence of injective $f : [0, 1) \times [0, 1) \to [0, 1)$ and injective $g : [0, 1) \to [0, 1) \times [0, 1)$ guarantees the existence of a bijection between $[0, 1)$ and $[0, 1) \times [0, 1)$.

2. **(6 points)** *A real number is called* transcendental *if it is not a root of any polynomial with integer coefficients. Prove that the set of transcendental numbers is uncountable.*

We shall start by showing that the set of numbers which are roots of polynomials is uncountable. Using the same techniques as seen in questions 1(b) and 1(c), we may assert that the polynomials with integer coefficients are denumerable: for any polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, let us put its roots in the class A_i if $|a_n| + |a_{n-1}| + \cdots + |a_0| = i$. Clearly there are a finite number of such polynomials (we could sloppily bound this number with $(2i + 1)^(2i + 1)$, which is ridiculously large but still finite) and since each polynomial has a finite number of roots, each A_i is finite, so we know that the $\bigcup_{i=0}^{\infty} A_i$ is denumerable (a.k.a. countably infinite) since we may inject $\mathbb{N}$ into a union of finite sets $A_1, A_2, \ldots$ by associating the first $|A_1|$ natural nubers with elements of A_1, the next $|A_2|$ with elements of A_2, and so forth.

Thus, the set of non-transcendental numbers is countable. If the set of transcendental numbers were also countable, their union would necessarily be countable; however, since $\mathbb{R}$ is in fact uncountable, we know this is not the case.

3. **(6 points)** *Prove that if* S *and* T *are denumerable sets, so is* $S \times T$.

Since S and T are denumerable, there are injections $f : S \to \mathbb{N}$ and $g : T \to \mathbb{N}$. Clearly we can craft a function $h : S \times T \to \mathbb{N} \times \mathbb{N}$ by letting $h((s,t)) = (f(s), g(t))$; clearly h is an injection, since in order for $h((s,t))$ to equal $h((s',t'))$ it must be the case that the ordered pairs $(f(s), g(t))$ and $(f(s'), g(t'))$ are equal, which happens only when $f(s) = f(s')$ and $g(t) = g(t')$. By injectivity of f and g, this necessitates that $s = s'$ and $t = t'$. Thus, $h((s,t)) = h((s',t'))$ only if $(s,t) = (s',t')$, so h is an injection.

To show denumerability of $S \times T$, however, we want an injection from $S \times T$ to $\mathbb{N}$, not one to $\mathbb{N} \times \mathbb{N}$. We shall thus make use of a canonical bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$: let $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $\phi((1,1)) = 1$, $\phi((1,2)) = 2$, $\phi((2,1)) = 3$, $\phi((3,1)) = 4$, $\phi((2,2)) = 5$, and so forth; specifically, $\phi(i-n, 1+n) = \frac{i(i+1)}{+}n$. This guarantees that all ordered pairs (a,b) with $a + b = i + 1$ are associated with numbers between the $(i-1)$th and ith triangular numbers. There are straitforward canonical presentations of why this is a bijection; since it is a bijection, $\phi \circ h$ is the desired injection from $S \times T$ into $\mathbb{N}$.

4. **(4 points)** *Show that given a set S and injective function $f : \mathcal{P}(S) \to \mathbb{N}$, S must be finite.*

   Suppose S is infinite; thus there is a surjection $g : S \to \mathbb{N}$. By diagonalization we shall show that there is no surjection $h : \mathbb{N} \to \mathcal{P}(S)$. Suppose, contrariwise, that there is such an h. Now we shall fabricate a set $A \in \mathcal{P}(S)$ which is guaranteed to not be in the image of h. Let $s \in A$ if and only if $s \notin h(g(s))$. Now we shall show that for each $n \in \mathbb{N}$, $A \neq h(n)$. For any given $n \in \mathbb{N}$, surjectivity of g guarantees that some $s_0 \in S$ is such that $g(s_0) = n$. Thus $s_0 \in A$ if and only if $s \notin h(n)$, so clearly $A \neq h(n)$, since they disagree on membership of s_0. Thus, $A \neq h(n)$ for any n, so A is not in the image of h, so h cannot be a surjeciton. Since there is no surjection from $\mathbb{N}$ to $\mathcal{P}(S)$, there is no injection from $\mathcal{P}(S)$ to $\mathbb{N}$.

5. **(4 point bonus)** *Let a "description" of a number be a finite string of letters that uniquely determines its value: for instance, "the positive square rrot of two" and "the positive root of the polynomial x squared minus two" are both descriptions for $\sqrt{2}$, and "the ratio of the circumference of a circle to its diameter" is a description for π. Prove that almost all real numbers do not have descriptions.*

   A "description consisting of n characters" is a string of n letters and spaces that makes sense in English; there are 27^n strings of such letters and spaces in total, of which only a few make sense, so clearly the set of numbers described by n-letter descriptions is finite; call that set A_n. As seen several times above, we can take the union of a countable collection of finite sets to get a countable set, so the set of describable numbers $\bigcup_{n=1}^{\infty} A_n$ is countable. Since the describable numbers are a countable set whereas $\mathbb{R}$ is uncountable, the describable numbers form an infinitesimal portion of the real numbers.

   From a philosophical point of view, this is rather unusual: having asserted that there are a "larger" number of real numbers than integers, we are in fact powerless to get a handle on the *kind* of numbers which make $\mathbb{R}$ a larger set — those numbers which contribute towards $\mathbb{R}$'s vastness are, as we just saw above, in fact incapable of being described with any number of words, and in spite of existing, are definitionally incomprehensible to any of our mathematical or expository faculties.

> Hay un concepto que es el corruptor y el desatinador de los otros. No hablo del mal cuyo limitado imperio es la ética; hablo del infinito. [There is a concept which is the corruptor and ruin of all others. I speak not of evil, whose realm is limited to ethics; I speak of the infinite.] —Jorge Luis Borges, "Avatares de la tortuga"