1. **(10 points)** *Let $S = \big\{\emptyset, \{1,2,3\}, \{2,4,6,8\}\big\}$. For each of the following criteria, either find a set satisfying those conditions, or explain why such a set does not exist.*

   - *Find $A \in S$ such that $|A| = 2$.*

     This set doesn't actually exist! If we were to select an *element* of S, then our only options are $\emptyset$, which is a zero-element set, $\{1,2,3\}$, which is a three-element set, or $\{2,4,6,8\}$, which is a four-element set. Thus, no element of S is a set of two elements.

   - *Find $B \subseteq S$ such that $|B| = 2$.*

     We can let B be a set containing any two elements of S; then by construction B is a subset of S and has two elements. There are three ways to do this, so we might let B be any of the three sets $\big\{\emptyset, \{1,2,3\}\big\}$, $\big\{\emptyset, \{2,4,6,8\}\big\}$, or $\big\{\{1,2,3\}, \{2,4,6,8\}\big\}$.

   - *Find $C \in \mathcal{P}(S)$ such that $|C| = 2$.*

     This is actually the *exact* same criterion as in the previous part, since being an element of $\mathcal{P}(S)$ and being a subset of S are the same concept. Thus, the same three possible values given above for B suffice for C.

   - *Find $D \subseteq \mathcal{P}(S)$ such that $|D| = 2$.*

     $\mathcal{P}(S)$ has eight elements, each of which is a subset of S. If we select a set containing two of them, then such a set is definitionally a subset of $\mathcal{P}(S)$ which contains two elements. There are in fact 28 different possible values for D; some good examples of possible values for D include $D = \big\{\emptyset, \{\emptyset\}\big\}$, $D = \big\{\emptyset, S\big\}$, or $D = \big\{\{\{1,2,3\},\{2,4,6,8\}\}, \{\{1,2,3\}\}\big\}$.

2. **(10 points)** *Find an infinite family of sets $A_1, A_2, A_3, A_4, \ldots$ such that*

$$\bigcup_{i=1}^{\infty} A_i = [-2, 2)$$

and

$$\bigcap_{i=1}^{\infty} A_i = (-1, 1].$$

Note that $[a, b)$ is notation for all real numbers between a and b including a but not including b.

This is rendered somewhat difficult by the inclusions of the endpoints. There are a number of ways to "cheat" your way to a solution. For instance, if we established $A_1 = [-2, 2)$ and $A_2 = (-1, 1]$, then $A_1 \cup A_2$ and $A_1 \cap A_2$ are already the desired values, and we just need to choose $A_3, A_4, \ldots$ so that they neither expand the union nor shrink the intersection. Letting $A_3 = A_4 = \cdots = [-1, 1]$, for instance, would work.

For a solution more in the spirit of the question, we might let $A_1 = [-2, 2)$, $A_2 = (-1, \frac{3}{2})$, $A_3 = (-1, \frac{4}{3})$, and so forth, with $A_i = (-1, \frac{i+1}{i})$. Then the union clearly is just A_1, since all the other sets lie within it, while the intersection will have a lower bound of -1 exclusive, like every set from A_2 on, and its upper bound will be 1 inclusive, since every A_i contains 1, but every number larger than 1 lies outside *some* A_i.

3. **(10 points)** *For statements P, Q, and R, show that $\big((P \to Q) \wedge (Q \to R)\big) \to (P \to R)$ is a tautology.*

The most straightforward approach to this statement is via a truth table.

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \wedge (Q \to R)$	$P \to R$	$\big((P \to Q) \wedge (Q \to R)\big) \to (P \to R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since the final column is true in all cases, it is a tautology. This particular tautological statement is called the *law of syllogism*, and forms a basic rule of inference.

4. **(10 points)** *Translate the statement $\exists A : \mathcal{P}(A) = \emptyset$ into words. Now write, in as easily-understood a form as possible, a negation of this statement in words, and also translate your negation back to symbols. Explain why this negation is true (from which we can determine that the original statement is false).*

We may read this statement as “There is an object (specifically, a set) A such that the power set of A is empty.” Bearing in mind that the negation of an existential statement is a universal one, we can write the negation of the above as “For every set A, the power set of A is nonempty.”, or, in symbols, $\forall A : \mathcal{P}(A) \neq \emptyset$. This is a statement we know to be true for various reasons; for instance, on a previous short problem, we noted that $A \in \mathcal{P}(A)$, and thus $\mathcal{P}(A)$ is nonempty, since it contains at least one element.

5. **(5 point bonus)** *Find as small a set A as possible such that $|A \cap \mathcal{P}(A)| \geq 2$. Exhibit that A and $\mathcal{P}(A)$ have at least two elements in common, and briefly defend your contention that it is the smallest such set.*

We want A and $\mathcal{P}(A)$ to have two elements in common; thus, we want there to be two things which are elements both of A and $\mathcal{P}(A)$; translating the concept of an element of $\mathcal{P}(A)$, we see that what we really want is two objects which are both elements and subsets of A. One easy freebie would be that if $\emptyset \in \mathcal{A}$, then $\emptyset$ would be both an element and subset of A. Getting a second such element, however, is a bit tricky. The easiest way to do it would be to make some x and $\{x\}$ both elements of A, so that $\{x\}$ would be both an element and subset of A. This gives us an easy three-element construction $A = \big\{\emptyset, x, \{x\}\big\}$ in which $A \cap \mathcal{P}(A) = \{\emptyset, \{x\}\}$. But we can actually do slightly better, by letting $x = \emptyset$ and reusing the element that's already there:

$$A = \big\{\emptyset, \{\emptyset\}\big\},$$

so that $A \cap \mathcal{P}(A) = \big\{\emptyset, \{\emptyset\}\big\}$. This is surely the smallest A could be, since in order for $A \cap \mathcal{P}(A)$ to have two elements, A itself would need to have 2 elements.

> “I know what you're thinking about,” said Tweedledum: “but it isn't so, nohow.” “Contrariwise,” continued Tweedledee, “if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic.”
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> —Lewis Carroll, *Through the Looking-Glass*