1. **(6 points)** Prove that for every real number $x > -1$ and positive integer $n$, it is the case that $(1+x)^n \geq 1 + nx$. (Hint: use induction on $n$)

2. **(10 points)** We learned about three properties of relations: reflexivity, symmetry, and transitivity. There are 8 possible combinations of these three properties (each could be present or absent). For *each* of the eight possibilities, find a set $S$ and a relation $R$ with that combination of properties, specifying which is which, of course. The relation can be described in words or symbols (e.g. "the relation $\leq$ on the set of real numbers"), or an explicit set of ordered pairs can be given.

3. **(10 points)** Suppose that you have a large collection of 4-cent and 9-cent stamps. Note that with these two stamps it is *impossible* to exactly make up a 23 cent postage. Prove using strong induction (or another method, if preferred) that every number of cents larger than 23 *can* be exactly made with these two types of stamps.

4. **(14 points)** Prove or disprove each of the two following statements:

   (a) **(7 points)** If $f : A \to B$ is an injective function and $g : B \to C$ is a surjective function, then $g \circ f$ is surjective.

   (b) **(7 points)** For functions $f : A \to B$ and $g : B \to C$, if $g \circ f$ is injective, then $f$ is injective.

5. **(5 point bonus)** Recall the recurrence, described in class, which is known as the *Catalan numbers*:

$$C_1 = 1$$
$$C_2 = C_1C_1 = 1$$
$$C_3 = C_2C_1 + C_1C_2 = 2$$
$$C_4 = C_3C_1 + C_2C_2 + C_1C_3 = 5$$
$$C_n = \sum_{i=1}^{n-1} C_{n-i}C_i = C_{n-1}C_1 + C_{n-2}C_2 + \cdots + C_2C_{n-2} + C_1C_{n-1} \text{ for } n > 1$$

   Prove that $C_n$ is odd if and only if $n$ is a power of 2.

6. **(5 point bonus)** Below is a proof that all horses are of the same color. Explain the flaw.

   **Proposition 1.** *Any finite set $S$ of horses are all of the same color.*

   *Proof.* We shall use induction on the size of the set. The base case $|S| = 1$ is trivial; a single horse is necessarily of the same color as itself (for simplicity, we treat "speckled" or suchlike as colors in their own right).

   For our inductive step, we consider a set $S$ of $k$ horses with $k \geq 2$, and may assume inductively that any set of $k-1$ horses are all the same color. Now let us select a horse $x$; by our inductive hypothesis, all the horses in the slightly smaller set $S - \{x\}$ are of the same color. Likewise, we select a horse $y$, and by our inductive hypothesis, all the horses in the set $S - \{y\}$ are of the same color. Now, if we select a horse from the overlap of $S - \{x\}$ and $S - \{y\}$, it must be the same color as the horses in *both* sets, so all the horses in $S - \{x\}$ and $S - \{y\}$ — which is to say, all the horses in $S$ — are the same color. $\square$

 **due November 9**