1. **(15 points)** *Let $A = \{\emptyset, 3, 4, \{10, 12\}\}$ and let $B = \{\{3\}, 4, \{10\}\}$. Find sets matching the following descriptions or, where such a set does not exist, explain why.*

   (a) $A \cap B$.

   Note that the only element A and B have in common is 4, so their intersection is $\boxed{\{4\}.}$ In particular, even though the numbers 3 and 10 appear in both enumerations, note that they are not identical elements between A and B: $3 \neq \{3\}$ and $\{10, 12\} \neq \{10\}$.

   (b) *A set $X \subset A$ such that $|X| = 1$.*

   A one-element subset of X is easy to find. There are in fact four different possible choices: $\boxed{\{\emptyset\}, \{3\}, \{4\}, \text{ or } \{\{10, 12\}\}.}$

   (c) *A set $Y \in A$ such that $|Y| = 1$.*

   A has no one-element sets as elements. It contains the zero-element set $\emptyset$ and the two-element set $\{10, 12\}$, but no one-lement sets.

2. **(12 points)** *Identify each of the following statements as a tautology, a contradiction, or neither. Show your work.*

   (a) $P \Leftrightarrow (Q \vee \neg P)$.

   As seen below, this is neither a tautology nor a contradiction.

P	Q	$\neg P$	$Q \vee \neg P$	$P \Leftrightarrow (Q \vee \neg P)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	F
F	F	T	T	F

   (b) $P \Rightarrow (Q \Rightarrow P)$.

   As seen below, this is a tautology.

P	Q	$Q \Rightarrow P$	$P \Rightarrow (Q \Rightarrow P)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

   (c) $(P \vee Q) \wedge (\neg P \vee \neg Q)$.

   As seen below, this is neither a tautology nor a contradiction.

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$(\neg P \vee \neg Q)$	$(P \vee Q) \wedge (\neg P \vee \neg Q)$
T	T	T	F	F	F	F
T	F	T	F	T	T	T
F	T	T	T	F	T	T
F	F	F	T	T	T	F

3. **(8 points)** *Prove or disprove: for any three sets A, B, and C, it is the case that $(A - B) - C \subseteq A - (B - C)$.*

   This is a true statement, which we shall prove.

Suppose $x \in (A-B)-C$; we shall endeavor to show that x is necessarily also an element of $A-(B-C)$. Since $x \in (A-B)-C$, it follows form the definition of set subtraction that $x \in A-B$ and $x \notin C$; applying set subtraction again, that first clause can be expanded to show that $x \in A$ and $x \notin B$. Since $x \notin B$, we know that $x \notin B-C$ (since membership in B is a necessary condition of membership in $B-C$). Then since $x \in A$ and $x \notin B-C$, it follows that $x \in A-(B-C)$.

4. **(8 points)** *Prove or disprove: for integers a and b, if $a^2 \nmid b^2$, then $a \nmid b$.*

   We shall prove the contrapositive, namely, that if $a \mid b$, then $a^2 \mid b^2$. Given that $a \mid b$, we know there must be an integer k such that $b = ka$. Thus $b^2 = (ka)^2 = k^2a^2$. Since k^2 is an integer, we may thus conclude that $a^2 \mid b^2$.

5. **(9 points)** *Prove that the identity $1^2+2^2+3^2+4^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ holds for all positive integers n.*

   We shall prove this by induction on n. In the base case $n=1$, note that 1^2 is indeed equal to $\frac{1(1+1)(2+1)}{6}$ (as both evaluate to 1).

   For our inductive step, we assume that for a specific k,

$$1^2+2^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$$

   and then, adding $(k+1)^2$ to each side, we perform arithmetic on the right side until we get the result we desire:

$$\begin{aligned}
1^2+2^2+\cdots+k^2+(k+1)^2 &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} \\
&= \frac{(k+1)\big[k(2k+1)+6(k+1)\big]}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)\big[(k+1)+1\big]\big[2(k+1)+1\big]}{6}
\end{aligned}$$

   which completes our inductive step.

6. **(15 points)** *For each of the following relations R on given sets S, determine whether each of the reflexive, symmetric, and transitive properties hold. Briefly justify your claims.*

   (a) $S = \mathcal{P}(\mathbb{N})$, *with R given by the criterion that $A \mathrel{R} B$ if and only if $A \cap B = \emptyset$.*

   This relation is $\boxed{\text{not reflexive}}$ since generally speaking $A \cap A \neq \emptyset$ (and, in fact, the only set A such that $A \mathrel{R} A$ would be the empty set itself; this relation is thus not only nonreflexive but very nearly antireflexive.

   This relation is $\boxed{\text{symmetric}}$ since disjunction of A and B is not dependent on the order in which the intersection is evaluated: $A \cap B$ and $B \cap A$ are the same set, and are either both empty or neither empty.

   This relation is $\boxed{\text{not transitive}}$ as can be seen by a simple example: if $A = \{1,2\}$, $B = \{3\}$, and $C = \{1,4\}$, we see that $A \mathrel{R} B$ and $B \mathrel{R} C$, but $A \not\mathrel{R} C$.

(b) *$S = \{1,2,3,4\}$, with $R = \{(1,1),(1,2),(2,1),(2,3),(3,2),(3,3),(3,4),(4,3)\}$.*

This relation is $\boxed{\text{not reflexive}}$ since $(2,2) \notin R$, or, alternatively, since $2 \not R\, 2$.

This relation is $\boxed{\text{symmetric}}$ since for each ordered pair $(x,y) \in R$, the ordered pair (y,x) is also in R.

This relation is $\boxed{\text{not transitive}}$ since $1\,R\,2$ and $2\,R\,3$, but $1 \not R\, 3$.

(c) *$S = \mathbb{N}$, with R given by the criterion that $a\,R\,b$ if and only if $a \le b^2$.*

This relation is $\boxed{\text{reflexive}}$ since for every natural number a, $a \le a^2$, and thus $a\,R\,a$.

This relation is $\boxed{\text{not symmetric}}$ since in general, if $a \le b^2$, it is not the case that $b \le a^2$: a specific counterexample might be, for instance, the case of $a = 1$ and $b = 2$.

This relation is $\boxed{\text{not transitive}}$ since, if $a \le b^2$ and $b \le c^2$, we may conclude that $a \le c^4$, but not that $a \le c^2$, and in fact we can find specific counterexamples: if $a = 5$, $b = 3$, and $c = 2$, it is indeed true that $5 \le 3^2$ and $3 \le 2^2$, but it is not true that $5 \le 2^2$.

7. **(18 points)** *Suppose that $f : A \to B$ is a function; let R be a relation on A such that for $x, y \in A$, the statement $x\,R\,y$ is true if and only if $f(x) = f(y)$.*

(a) *Using no specific knowledge about f except that it is a function, prove that R is an equivalence relation.*

For any element a of A, $f(a)$ is a specific element of B, which is definitionally equal to itself. Thus $f(a) = f(a)$, so $a\,R\,a$, guaranteeing reflexivity.

For any elements x and y of A, $f(x)$ and $f(y)$ are elements of B. Symmetry of equality guarantees that $f(x) = f(y)$ if and only if $f(y) = f(x)$, and thus $x\,R\,y$ and $y\,R\,x$ have identical truth values, asserting symmetry.

For any elements x, y, and z of A, taking the premise that $x\,R\,y$ and $y\,R\,z$, we thus know that $f(x) = f(y)$ and $f(y) = f(z)$. Using transitivity of equality, we see that $f(x) = f(z)$, so $x\,R\,z$, demonstrating transitivity.

(b) *If f is injective, what can you say about the equivalence classes of R?*

If f is injective, then $f(x) = f(y)$ only when $x = y$; thus R is identical to the relation of equality. Since any element of A is equal only to itself, each element of A will lie in its own equivalence class.

(c) *If f is constant (i.e. there is a $b \in B$ such that for every $a \in A$, $f(a) = b$), what can you say about the equivalence classes of R?*

If f is constant mapping every element of A to a specific b, then for any $x, y \in A$, it will be the case that $f(x) = b = f(y)$, so $x\,R\,y$, making R a universally-true relation. Since an equivalence class consists of all elements which are related to each other, all the elements of A will lie in a single equivalence class.

8. **(15 points)** *Identify each of the following functions with the stated domains and codomains, as injective, surjective, bijective, or none of the above. Briefly justify your claim.*

(a) *$f : [-2,2] \to [0,2]$ given by $f(x) = |x|$.*

This function is $\boxed{\text{surjective}}$ since every element of the codomain is the image of some element of the domain: for $0 \le y \le 2$, it is easy to find an element of $[-2,2]$ whose absolute value is y; specifically, $|y| = y$.

This function is $\boxed{\text{not injective}}$ since, for example, $f(-1) = f(1) = 1$.

(b) $g : \mathbb{N} \to \mathbb{N}$ *given by the rule that* $g(n)$ *is equal to the largest prime factor of* n*, with the special case that for* $n < 2$, $g(n) = n$.

This function is $\boxed{\text{not surjective}}$ since no composite numbers are mapped to by construction: for instance, there is no n such that $g(n) = 4$.

This function is $\boxed{\text{not injective}}$ since many integers have the same largest prime factor, for instance, $g(3) = g(6) = g(9) = g(12) = g(18) = 3$.

(c) $h : \mathbb{Z} \to \mathbb{Q}$ *given by* $h(n) = n$.

This function is $\boxed{\text{not surjective}}$ since every element of the domain is simply mapped to itself, but the codomain contains many values not in the domain: for instance, there is no n such that $h(n) = \frac{1}{2}$, since $\frac{1}{2} \notin \mathbb{Z}$.

This function is $\boxed{\text{injective}}$ since $h(n) = n$ and $h(m) = m$, and thus if $h(n) = h(m)$, then $n = m$.

Problems worthy
of attack
prove their worth
by hitting back.

—Piet Hein, "Problems"