1. **(12 points)** *Give examples of sets satisfying the following conditions, or explain why they cannot be met:*

   (a) *sets A, B, and C such that $A \subseteq B \subsetneq C$.*

   Any sets such that every element of A is an element of B and C, every element of B is an element of C, and $B \neq C$ will suffice. A minimal example is $A = \emptyset$, $B = \emptyset$, $C = \{1\}$. A more natural example might be $A = \{1\}$, $B = \{1, 2\}$, $C = \{1, 2, 3\}$.

   (b) *sets R, S, and T such that $R \in S$, $S \in T$, and $R \notin T$.*

   T will need to be a set containing at least one set containing at least one set, so we'll need set-containment nested at least 3 levels deep; S will be the set so contained, and R will be the set contained in it. Unless we do something particularly unusual, the condition $R \notin T$ should end up being satisfied easily. A minimal example is $R = \emptyset$, $S = \{\emptyset\}$, $T = \big\{\{\emptyset\}\big\}$. A more natural example might be $R = \{1, 2\}$, $S = \big\{\{1, 2\}, 3\big\}$, $T = \Big\{0, \big\{\{1, 2\}\big\}\Big\}$.

   (c) *sets X, Y, and Z such that $X \in Y$ and $Y \subsetneq Z$.*

   Y needs to be a set containing another set, which is X; Z simply needs to have all the elements of Y and at least one more. A minimal example is $X = \emptyset$, $Y = \{\emptyset\}$, and $Z = \{\emptyset, 1\}$. A more natural example might be $X = \{1\}$, $Y = \big\{\{1\}, 2, 3\big\}$, $Z = \big\{\{1\}, 2, 3, 4\big\}$.

2. **(12 points)** *Let $A_i = \{i, i+1, i+2, \ldots, i+100\}$, so that each set A_i has 100 elements which are positive integers. Calculate the results of the following indexed set operations:*

   (a) $\displaystyle\bigcap_{i=1}^{100} A_i$.

   Note that $A_1 = \{1, 2, \ldots, 101\}$ and $A_{100} = \{100, 101, \ldots, 200\}$. Since $A_1 \cap A_{100} = \{100, 101\}$, this indexed intersection of 100 sets, two of which are A_1 and A_{100}, can contain no *more* than the two elements 100 and 101. And, in fact, it is easy to see that 100 and 101 are in fact elements of every single one of $A_1, A_2, \ldots, A_{100}$, so $\displaystyle\bigcap_{i=1}^{100} A_i = \{100, 101\}$

   (b) $\displaystyle\bigcup_{i=1}^{100} A_i$.

   Using the known values of A_1 and A_{100} above, we can see that $A_1 \cup A_{100} = \{1, 2, \ldots, 200\}$, so the indexed union of 100 sets including these two must contain *at least* these 200 elements. In fact, since every element of $A_1, A)2, \ldots, A_{100}$ lies in this set, this is a complete accounting of the union, and $\displaystyle\bigcup_{i=1}^{100} A_i = \{1, 2, \ldots, 200\}$.

   (c) $\displaystyle\bigcap_{i=1}^{102} A_i$.

   We may note that A_1 and A_{102} are disjoint, since the former only contains elements less than 101 and the latter only elements greater than 101. Since $A_1 \cap A_{102} = \emptyset$, we know *any* intersection including these two sets (even together with 100 other sets, as in this case) will be empty.

   (d) $\displaystyle\bigcup_{i=1}^{\infty} A_i$.

   Every positive integer lies in *some* A_i; in particular, each $i \in A_i$, so the union of all the sets A_i includes every positive integer. Since the sets A_i by construction contain only positive itnegers, we see that $\bigcup_{i=1}^{\infty} A_i$ is the set containing every positive integer, which could be denoted $\{1, 2, 3, 4, \ldots\}$ or simply $\mathbb{N}$.

3. **(12 points)** *Write out truth tables for each of the following statements (you may write them all in one truth table, if you wish):*

   (a) $(P \wedge Q) \rightarrow Q$.

P	Q	$P \wedge Q$	$(P \wedge Q) \rightarrow Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

   (b) $P \wedge \neg(Q \vee P)$.

P	Q	$Q \vee P$	$\neg(Q \vee P)$	$P \wedge \neg(Q \vee P)$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

   (c) $(\neg P) \leftrightarrow (P \vee \neg Q)$.

P	Q	$\neg P$	$\neg Q$	$P \vee \neg Q$	$(\neg P) \leftrightarrow (P \vee \neg Q)$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	F
F	F	T	T	T	T

4. **(6 points)** *Determine the converse of the true statement "If $S \subseteq T$, then $|S| \leq |T|$." Is the converse itself true? Either briefly justify your statement or provide a counterexample.*

   The converse is "If $|S| \leq |T|$, then $S \subseteq T$." This is demonstrably false, as can be seen with the counterexample $S = \{1\}$ and $T = \{2, 3\}$. Here $|S| = 1$ and $|T| = 2$, so $|S| \leq |T|$, but $S \not\subseteq T$, since $1 \in S$ and $1 \notin T$.

5. **(6 points)** *Write the negation of "There is an element n of S such that $n^2 \in S$" as a quantified statement (using a universal or existential quantifier, either in words or in symbols).*

   The negation is: "for all elements n of S, $n^2 \notin S$", or alternatively $\forall n \in S : n^2 \notin S$.

6. **(12 points)** *Prove that for an integer n, if $5n^3 \not\equiv 0 \pmod 3$, then $n \not\equiv 0 \pmod 3$.*

   Proof. We restate the proposition as its contrapositive: for an integer n, we shall prove that if $n \equiv 0 \pmod 3$, then $5n^3 \equiv 0 \pmod 3$. By our premise, $3 \mid (n - 0)$, or in other words, $3 \mid n$. It is known that if $k \mid n$, then $k \mid cn$ for any integer c, so $3 \mid n \cdot 5n^2$; arithmetically, we can rewrite that as $3 \mid (5n^3 - 0)$, so $5n^3 \equiv 0 \pmod 3$. □

7. **(12 points)** *Let a and b be nonzero integers. Prove that if $a \mid b$ and $b \mid a$, then either $a = b$ or $a = -b$.*

Proof. Interpreting our premises in terms of the definition of divisibility, we know there are integers k and ℓ such that $b = ka$ and $a = \ell b$. Thus, $b = k(\ell b)$; since b is nonzero we may cancel it from both sides of the equality to conclude that $k\ell = 1$, Since the only integer factors of 1 are 1 and -1, it follows that either $k = \ell = 1$ or $k = \ell = -1$; in the former case, $a = b$, and in the latter, $a = -b$. □

One can alternatively prove the contrapositive, although it's a bit tricky:

Proof. We shall instead prove the contrapositive: if $a \neq b$ and $b \neq a$, then either $a \nmid b$ or $b \nmid a$. Our premise that $a \neq \pm b$ can be more succinctly stated as asserting that $|a| \neq |b|$. Without loss of generality we could specify $|a| > |b|$, and from this we shall seek to show that $a \nmid b$. Noting that $\left|\frac{b}{a}\right| = \frac{|b|}{|a|} < 1$, and that since $b \neq 0$, $\left|\frac{b}{a}\right| > 0$; thus $0 < \left|\frac{b}{a}\right| < 1$, so that $\frac{b}{a}$ is an element of $(-1, 0) \cup (0, 1)$, which notably contains no integers. Thus there is no integer k such that $b = ka$, so $a \nmid b$. □