1. **(12 points)** *Let $S = \{\{2,3\}, 2, \{1,2,3,4\}\}$. For each of the following descriptions, either produce a set matching the description or explain briefly why such a set doesn't exist.*

   (a) *A set A of 4 elements such that $A \subseteq S$.*
   Since S has only 3 elements (the set $\{2,3\}$, the number 2, and the set $\{1,2,3,4\}$), none of its subsets can have more than 3 elements; thus no such set A exists.

   (b) *A set B of 4 elements such that $B \subseteq \mathcal{P}(S)$.*
   A 4-element subset of $\mathcal{P}(S)$ must be a collection of 4 elements of $\mathcal{P}(S)$; in other words, it must be a collection of 4 subsets of S. S has 8 distinct subsets, and we can choose any 4 we like. Here's an example:

$$B = \Big\{\emptyset, \{\{1,2,3,4\}\}, \{2\}, \{\{1,2,3,4\}, 2\}\Big\}$$

   (c) *A set C of 4 elements, such that $C \in S$.*
   S has a 4-element subset as an element, namely $\{1,2,3,4\}$.

   (d) *A set D of 4 elements, such that $D \in \mathcal{P}(S)$.*
   $D \in \mathcal{P}(S)$ means that $D \subseteq S$; part (a) explained why such a set does not exist.

2. **(12 points)** *Identify each of the following statements as a tautology, a contradiction, or neither. Show your work.*

   (a) $(P \wedge Q) \Rightarrow P$.
   As seen below, this is a tautology.

P	Q	$P \wedge Q$	$(P \wedge Q) \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

   (b) $(P \Rightarrow Q) \Leftrightarrow (P \vee Q)$.
   As seen below, this is a neither a tautology nor a contradiction.

P	Q	$P \Rightarrow Q$	$P \vee Q$	$(P \Rightarrow Q) \Leftrightarrow (P \vee Q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

   (c) $(P \wedge \neg Q) \wedge (P \wedge Q)$.
   As seen below, this is a contradiction.

P	Q	$\neg Q$	$P \wedge \neg Q$	$P \wedge Q$	$(P \wedge \neg Q) \wedge (P \wedge Q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	F	F
F	F	T	F	F	F

3. **(6 points)** *Prove or disprove: if n is an integer, then $n^3 - n$ is even.*

Proof. We may best approach this problem by dividing it into cases depending on the parity of n:

Case I: n is even. Thus there is an integer k such that $n = 2k$, and then $n^3 - n = (2k)^3 - 2k = 8k^3 - 2k = 2(4k^3 - k)$; since $4k^3 - k$ is an integer, this expression is clearly even.

Case II: n is odd. Thus there is an integer k such that $n = 2k + 1$, and then $n^3 - n = (2k+1)^3 - (2k+1) = 8k^3 + 12k^2 + 6k + 1 - (2k+1) = 8k^3 + 12k^2 + 4k = 2(4k^3 + 6k^2 + 2k)$; since $4k^3 + 6k^2 + 2k$ is an integer, this expression is clearly even. □

4. **(6 points)** *Prove or disprove: if n is a positive integer, then $3 \mid (2n^2 + 1)$.*

Disproof. This statement is easily shown to be false by the counterexample $n = 3$: $2 \cdot 3^2 + 1 = 19$, which is not divisible by 3. □

5. **(6 points)** *Prove that if A, B, and C are sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$.*

Proof. We shall prove this statement by contradiction (note: it is possible to approach this problem other ways). Let us suppose, contrariwise, that there are A, B, and C such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, but that $B \neq C$ nonetheless. In order for two sets to be unequal, one of them must contain an element the other does not; we may without loss of generality assert that B contains an element not in C; let us call it x, so that $x \in B$ but $x \notin C$. At this point there are two possibilities:

Case I: $x \in A$. Then since $x \in B$, $x \in A \cap B$, but since $x \notin C$, $x \notin A \cap C$. Thus $A \cap B \neq A \cap C$, contradicting our premise.

Case II: $x \notin A$. Then since $x \in B$, $x \in A \cup B$, but since $x \notin C$, $x \notin A \cup C$. Thus $A \cup B \neq A \cup C$, contradicting our premise. □

6. **(6 points)** *Let $a_1 = 1$ and for $n > 1$, let $a_n = \sqrt{1 + a_{n-1}}$. Prove that for all n, $1 \leq a_n < 2$.*

Proof. We prove this result by induction on n. Clearly in the case $n = 1$ the stated result holds: since $a_1 = 1$, it is the case that $1 \leq a_1 < 2$.

We now assume inductively that for a particular k, $1 \leq a_k < 2$, and seek to prove that $1 \leq a_{k+1} < 2$. Sicne $a_{k+1} = \sqrt{1 + a_k}$, it is easy to achieve the desired inequality via arithmetic on our inductive hypothesis:

$$\begin{aligned} 1 \leq a_k &< 2 \\ 2 \leq a_k + 1 &< 3 \\ 1 \leq \sqrt{2} \leq \sqrt{a_k + 1} &< \sqrt{3} < 2 \\ 1 \leq \sqrt{a_k + 1} &< 2 \\ 1 \leq a_{k+1} &< 2 \end{aligned}$$

□

7. **(12 points)** *For each of the following relations R on given sets S, determine whether each of the reflexive, symmetric, and transitive properties hold. Briefly justify your claims.*

   (a) $S = \{1, 2, 3, 4, 5\}$, $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1), (3, 3), (4, 5), (5, 4), (5, 5)\}$.

   R is clearly nonreflexive since $(2, 2)$ and $(4, 4)$ are not in R.

   R is symmetric as can be seen that for each non-reflexive pair present in R (i.e. $(1, 2)$, $(1, 3)$, and $(4, 5)$) the reverse pair is also present.

   R is clearly nontransitive since $(2, 1)$ and $(1, 3)$ are in R, but $(2, 3)$ is not.

   (b) $S = \mathbb{Z}$, *with R given by the criterion that $a\ R\ b$ if and only if $a + b$ is divisible by 3.*

   R is clearly nonreflexive, since $1 + 1$ is not divisible by 3, so $1 \not R\ b$.

   R is symmetric, since $a + b$ and $b + a$ are equal, so either both of them or neither of them are divisible by 3.

   R is clearly nontransitive, since $1\ R\ 2$ and $2\ R\ 4$, but $1 \not R\ 4$.

   (c) $S = \mathbb{R}$, *with R given by the criterion that $a\ b$ if and only if $a - b$ is a non-negative integer.*

   R is reflexive, since for every $x \in \mathbb{R}$, $x - x = 0$ which is indeed a non-negative integer, so $x\ R\ x$.

   R is not symmetric, since, for instance $3\ R\ 2$ because $3 - 2$ is a non-negative integer, but $2 \not R\ 3$ because $2 - 3$ is negative.

   R is transitive: if $a, b, c \in \mathbb{R}$ are such that $a\ R\ b$ and $b\ R\ c$, then it must be the case that $a - b = n$ and $b - c = m$ for non-negative integers n and m. Then $(a - b) + (b - c) = n + m$; the right side of this equation is clearly a non-negative integer, while the left side is simply $a - c$, so $a - c$ is a non-negative integer and thus $a\ R\ c$.

8. **(6 points)** *Prove or disprove: for sets A, B, C, and D, if $|A| = |C|$ and $|B| = |D|$, then $|A \cup B| = |C \cup D|$.*

   Disproof. There are several finite examples; whenever $A \cap B$ and $C \cap D$ are different sizes, this statement will be false. One easy example is $A = B = C = \{a\}$ and $D = \{b\}$. Then $|A \cup B| = |\{a\}| = 1$ while $|C \cup D| = |\{a, b\}| = 2$. □

9. **(12 points)** *Identify each of the following functions with the stated domains and codomains, as injective, surjective, bijective, or none of the above. Briefly justify your claim.*

   (a) $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = e^x$.

   This function is injective, since e^x increases throughout (so if $x < y$, then $e^x < e^y$), but is not surjective, since there is no real number x such that $e^x = 0$.

   (b) $g : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ *given by the rule that $g(S)$ is the smallest element of S, with the special-case rule $g(\emptyset) = 1$.*

   This function is surjective, since any number $n \in \mathbb{N}$ can be determined to lie in its image because $g(\{n\}) = n$. It is not injective, since, for instance, $g(\{1, 2\}) = g(\{1, 3\}) = 1$.

   (c) $h : \mathbb{Z} \to \mathbb{Z}$ *given by* $h(n) = n^2$.

   This function is not injective, since, for instance, $h(-2) = h(2) = 4$. It is also not surjective, since there is no element n of $\mathbb{Z}$ such that $h(n) = 2$.