1. **(12 points)** *A bar of iron which has been heated to 1400°F is taken from the furnace into a 100°F metalworking studio. After 5 minutes it has cooled to 800°F.*

   (a) **(5 points)** *Produce a function $T(t)$ modeling the bar's temperature t minutes after removal from the furnace.*

   We know that this problem is modeled by Newton's Law of Cooling with an ambient temperature of 100°F, so our temperature model will be $T(t) = 100 + Ce^{-kt}$; it remains only to find C and k to have a final model.

   Since the bar has a temperature of 1400°F immediately upon removal from the furnace, $T(0) = 1400$. Evaluating the left side of this equation, we find that $100 + Ce^0 = 1400$; thus $C = 1300$.

   Since the bar has a temperature of 800°F five minutes later, we know that $T(5) = 800$. Expanding $T(5)$, we find that:

$$\begin{aligned} 100 + 1300e^{-k\cdot 5} &= 800 \\ 1300e^{-5k} &= 700 \\ e^{-5k} &= \frac{7}{13} \\ -5k &= \ln\frac{7}{13} \\ k &= \frac{-\ln\frac{7}{13}}{5} \end{aligned}$$

   Assembling this value of k into our equation, we find that

$$T(t) = 100 + 1300e^{\frac{\ln\frac{7}{13}}{5}t}$$

   (b) **(3 points)** *How quickly is the bar's temperature changing immediately upon removal from the furnace?*

   The time of the bar's removal from the furnace is definitionally time 0; thus the speed of the bar's cooling at that time is $T'(0)$. From the value of $T(t)$ above, we can easily compute $T'(t)$:

$$T'(t) = 1300\frac{\ln\frac{7}{13}}{5}e^{\frac{\ln\frac{7}{13}}{5}t}$$

   so $T'(0) = 1300\frac{\ln\frac{7}{13}}{5}e^0 = 260\ln\frac{7}{13}$. This is approximately -161, signifying that the bar is cooling (dropping in temperature) by 161 degrees per minute.

   (c) **(4 points)** *The metal can be worked as long as it is hotter than 1000°F. How soon after the bar is removed from the furnace does it become unworkable?*

   Since the metal becomes unworkable when $T(t) = 1000$, we want to find the value of t satisfying that equation:

$$\begin{aligned}
100 + 1300e^{\frac{\ln \frac{7}{13}}{5}t} &= 1000 \\
1300e^{\frac{\ln \frac{7}{13}}{5}t} &= 900 \\
e^{\frac{\ln \frac{7}{13}}{5}t} &= \frac{9}{13} \\
\frac{\ln \frac{7}{13}}{5}t &= \ln \frac{9}{13} \\
t &= \frac{5 \ln \frac{9}{13}}{\ln \frac{7}{13}} \approx 3 \text{ minutes}
\end{aligned}$$

2. **(10 points)** *If* $f(r) = \sin(2r^3 \ln r)$, *calculate* $f'(r)$.

   Initial analysis suggests that we will be using both the chain rule (since $f(r)$ consists of the sine of a complicated expression), and the product rule (since that complicated expression is itself a product. We may define $u = 2r^4 \ln r$, and then:

$$\begin{aligned}
f'(r) &= \frac{d}{dr} \sin(u) \\
&= \frac{du}{dr}\frac{d}{du} \sin(u) \\
&= \left(\frac{d}{dr}(2r^3 \ln r)\right) \cos(u) \\
&= \left(\left(\frac{d}{dr}2r^3\right) \ln r + 2r^3 \frac{d}{dr} \ln r\right) \cos(u) \\
&= \left(6r^2 \ln r + 2r^3 \cdot \frac{1}{r}\right) \cos(2r^4 \ln r) \\
&= \left(6r^2 \ln r + 2r^2\right) \cos(2r^4 \ln r)
\end{aligned}$$

3. **(8 points)** *Estimate the following values using appropriate linear approximations.*

   (a) **(4 points)** $\sqrt[3]{7.994}$.

   We consider the function $f(x) = \sqrt[3]{x}$, whose derivative is $\frac{1}{3x^{2/3}}$. For x close to 8 (as 7.994 is), we can use the linear approximation:

$$f(x) \approx f(8) + (x - 8)f'(8)$$

   Since $f(8) = 2$ and $f'(8) = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{12}$, it follows that

$$f(7.994) \approx 2 - 0.006 \cdot \frac{1}{12} = 1.9995$$

   For purposes of comparison, the actual value of $\sqrt[3]{7.994}$ is around 1.999499875.

   (b) **(4 points)** $(-2.004)^5$.

   We consider the function $f(x) = x^5$, whose derivative is $f'(x) = 5x^4$. For x close to -2 (as -2.004 is), we can use the linear approximation:

$$f(x) \approx f(-2) + (x + 2)f'(-2)$$

Since $f(-2) = -32$ and $f'(-2) = 80$, it follows that

$$f(-2.004) \approx -32 - 0.004(80) = -32.32$$

For purposes of comparison, the actual value of $(-2.004)^5$ is around -32.3212826.

4. **(15 points)** *Hiro is motorcycling west at 60 mph from a point 120 miles east of the Black Sun, while Raven is 50 miles north of the Black Sun, driving northwards at 30 mph.*

   (a) **(12 points)** *Are they growing closer together or further apart, and at what speed are they doing so?*

   Let Hiro's distance to the west of the Black Sun in miles be x; let Raven's distance to the north of the Black Sun in miles be y. Let s be the distance between them, and let t be time in hours. Since Hiro is driving *towards* the Sun, x is decreasing at 60 miles per hour; that is, $\frac{dx}{dt} = -60$; Raven is driving away, so y is increasing at 30 miles per hour, or $\frac{dy}{dt} = 30$. Furthermore, Hiro, the Black Sun, and Raven form a right triangle with hypotenuse of s, so by the Pythagorean Theorem

$$x^2 + y^2 = s^2$$

   Since we want $\frac{ds}{dt}$, we differentiate this expression with respect to t to get:

$$\begin{aligned} 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 2s\frac{ds}{dt} \\ \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2s} &= \frac{ds}{dt} \\ \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{s} &= \frac{ds}{dt} \end{aligned}$$

   In the situation given here, $x = 120$ and $y = 50$ are given to us, and $s = \sqrt{120^2 + 50^2} = 130$, so $\frac{ds}{dt} = \frac{120(-60)+50(30)}{130} = \frac{-570}{13}$. Thus the rivals are approaching each other (i.e. s is decreasing) at $\frac{570}{13}$ miles per hour.

   (b) **(3 points)** *In an hour, will they be growing closer together or further apart, and at what speed will they be doing so?*

   In the situation given here, $x = 120 - 60 = 60$ and $y = 50 + 30 = 80$ are given to us, and $s = \sqrt{60^2 + 80^2} = 100$, so $\frac{ds}{dt} = \frac{60(-60)+80(30)}{100} = -12$. Thus the rivals are approaching each other (i.e. s is decreasing) at -12 miles per hour.

5. **(10 points)** *Find* $\frac{d}{dt}\frac{e^{7t-3}}{\arctan(t^2)}$.

Initial analysis suggests that we will be using both the quotient rule (since this expression is a quotient, on its outermost level), and the chain rule (since the arguments of the exponential and arctangent functions are themselves complicated expressions). We may pre-emptively define $u = 7t - 3$ and $v = t^2$, and then:

$$\begin{aligned}
\frac{d}{dt}\frac{e^u}{\arctan(v)} &= \frac{\arctan(v)\left(\frac{d}{dt}e^u\right) - e^u\frac{d}{dt}\arctan v}{\arctan(v)^2} \\
&= \frac{\arctan(v)\left(\frac{du}{dt}\frac{d}{du}e^u\right) - e^u\frac{dv}{dt}\frac{d}{dv}\arctan v}{\arctan(v)^2} \\
&= \frac{\arctan(v)\left(7e^u\right) - e^u\cdot 2t\cdot\frac{1}{1+v^2}}{\arctan(v)^2} \\
&= \frac{7\arctan(t^2)e^{7t-3} - \frac{2te^{7t-3}}{1+(t^2)^2}}{\arctan(t^2)^2}
\end{aligned}$$

6. **(15 points)** *The conchoid of de Sluze is a curve satisfying the equation* $(x-1)(x^2+y^2) = 4x^2$.

   (a) **(12 points)** *Find a formula for* $\frac{dy}{dx}$ *on this curve.*
   We differentiate both sides with respect to x, and use the product and chain rule as appropriate:

$$\begin{aligned}
\frac{d}{dx}\left((x-1)(x^2+y^2)\right) &= \frac{d}{dx}4x^2 \\
\left(\frac{d}{dx}(x-1)\right)(x^2+y^2) + (x-1)\left(\frac{d}{dx}(x^2+y^2)\right) &= 8x \\
1\cdot(x^2+y^2) + (x-1)(2x+2yy') &= 8x \\
x^2+y^2+(x-1)(2x)+(x-1)2yy' &= 8x \\
(x-1)2yy' &= 8x - x^2 - y^2 - (x-1)(2x) \\
y' &= \frac{10x-3x^2-y^2}{2(x-1)y}
\end{aligned}$$

   (b) **(3 points)** *Find the equation of the tangent line to the curve at* $(3,3)$.
   At the specific values $x = 3$, $y = 3$, we see from the above that

$$\frac{dy}{dx} = \frac{10\cdot 3 - 3\cdot 3^2 - 3^2}{2(3-1)3} = \frac{-6}{12} = \frac{-1}{12}$$

   Thus we have a line of the form $y = \frac{-x}{2} + b$; plugging in $(3,3)$ we can solve for b:

$$\begin{aligned}
3 &= \frac{-3}{2} + b \\
\frac{9}{2} &= b
\end{aligned}$$

   so $y = \frac{-x+9}{2}$ is the tangent line.

7. **(20 points)** *Answer the following derivative-related questions.*

(a) **(6 points)** *If* $f(x) = (x^2 - 4x)\arcsin(x)$, *find* $f'(x)$.
This is an application of the product rule:

$$\frac{d}{dx}\left((x^2-4x)\arcsin x\right) = \frac{d}{dx}\left(x^2-4x\right)\arcsin x + (x^2-4x)\frac{d}{dx}\arcsin x$$
$$= (2x-4)\arcsin x + (x^2-4x)\cdot\frac{1}{\sqrt{1-x^2}}$$

(b) **(8 points)** *Find* $\frac{d}{dt}\left(\ln(\cos t)\right)^5$.
This is an application of the chain rule in two stages, in which we shall let $u = \ln(\cos t) = \ln v$, and $v = \cos t$:

$$\begin{aligned}\frac{d}{dt}\left(\ln(\cos t)\right)^5 &= \frac{d}{dt}u^5\\ &= \frac{du}{dt}\frac{d}{du}u^5\\ &= \left(\frac{d}{dt}\ln v\right)\frac{d}{du}u^5\\ &= \frac{dv}{dt}\left(\frac{d}{dv}\ln v\right)\frac{d}{du}u^5\\ &= \sin t\cdot\frac{1}{v}\cdot 5u^4\\ &= \frac{5\sin t\left(\ln(\cos t)\right)^4}{\cos t}\end{aligned}$$

(c) **(6 points)** *If* $y = \frac{e^x}{x^3-\sqrt{x}}$, *find* $\frac{dy}{dx}$.
This is an application of the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\frac{e^x}{x^3-\sqrt{x}}\\ &= \frac{(x^3-\sqrt{x})\left(\frac{d}{dx}e^x\right) - e^x\frac{d}{dx}(x^3-\sqrt{x})}{\left(x^3-\sqrt{x}\right)^2}\\ &= \frac{(x^3-\sqrt{x})e^x - e^x(3x^2-\frac{1}{2\sqrt{x}})}{\left(x^3-\sqrt{x}\right)^2}\end{aligned}$$

8. **(10 points)** *Find an equation of the tangent line to the curve* $y = \frac{x^2+3}{x^2+x+1}$ *at* $(2, 1)$.
Using the quotient rule,

$$\frac{dy}{dx} = \frac{(x^2+x+1)(2x)-(x^2+3)(2x+1)}{(x^2+x+1)^2}$$

and specifically when $x = 2$, $\frac{dy}{dx} = \frac{7\cdot4-7\cdot5}{7^2} = \frac{-1}{7}$, so the desired line has slope $\frac{-1}{7}$, and thus equation $y = \frac{-x}{7} + b$. Since it passes through the point $(2, 1)$ we may specifically determine that $1 = \frac{-2}{7} + b$, and thus that $b = \frac{9}{7}$, for an equation $y = \frac{-x+9}{7}$ of the tangent line.